New Bounds on the Maximal Error Exponent for Multiple-Access Channels

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Abstract

The problem of bounding the reliability function of a multiple-access channel (MAC) is studied. An upper bound on the minimum Bhattacharyya distance between codeword pairs is derived. For a certain large class of two-user discrete memoryless (DM) MAC, a lower bound on the maximal probability of decoding error is derived as a consequence of the upper bound on Bhattacharyya distance. Further, an upper bound on the average probability of decoding error is studied. It is shown that the corresponding upper and lower bounds have a similar structure. Using a conjecture about the structure of the multi-user code, a tighter lower bound for the maximal probability of decoding error is derived and is shown to be tight at zero rates.

I. Introduction

An interesting problem in network information theory is to determine the minimum probability of error which can be achieved on a discrete memoryless (DM), multiple-access channel (MAC). Ahlswede [1] and Liao [2] studied the capacity region for MAC. Later, stronger versions of their coding theorem, giving exponential upper and lower bounds on the error probability, were derived by numerous other authors. Slepian and Wolf [3], Dyachkov [4], Gallager [5], Pokorny and Wallmeier [6], Liu and Hughes [7], and Nazari et al. [8] all studied *upper* bounds on the average probability of decoding error. Haroutunian [9] derived a *lower* bound on the optimal average error probability. Nazari et al. [10] derived a tighter lower bound that explicitly captures the separation of the encoders in the MAC. However, the bound in [10] is only valid for the maximal and not the average error probability.

In this paper, we derive a new lower bound on the maximal error probability for MAC. This bound is derived by establishing a link between minimum Bhattacharyya distance and maximal probability of decoding error; then, the upper bound on Bhattacharyya distance is used to infer the lower bound on probability of decoding error. Also, by the method of expurgation [8], an upper bound on the average probability of decoding error is derived. At zero rate pair, the upper and lower bounds have a similar structure, however, they may not be equal. By using a

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conjecture about the structure of the code, we derive another bound on the Bhattacharrya distance, which results in a tighter lower bound on the maximal probability of decoding error. At zero rate pair, this bound is tight, i.e., it is asymptotically equal to the upper bound.

The paper is organized as follows. Some preliminaries are introduced in section II. The main result of the paper, which is an upper bound on the reliability function of the channel, is obtained in section III. In section IV, a lower bound on the reliability function is developed and compared with the result of section III. Finally, in section V, a conjecture about the structure of all possible codes is proposed, and, based on the conjecture, another upper bound on the reliability function of the channel is obtained. It is shown that this bound is always asymptotically tight at zero rate pair.

II. PRELIMINARIES

For any alphabet $\mathcal{X}$, $\mathcal{P}(\mathcal{X})$ denotes the set of all probability distributions on $\mathcal{X}$. The *type* of a sequence $\mathbf{x} = (x_1, ..., x_n) \in \mathcal{X}^n$ is the distributions $P_{\mathbf{x}}$, on $\mathcal{X}$, defined by:

$$P_{\mathbf{x}}(x) \triangleq \frac{1}{n} N(x|\mathbf{x}), \qquad x \in \mathcal{X}, \tag{1}$$

where $N(x|\mathbf{x})$ denotes the number of occurrences of x in $\mathbf{x}$. Let $\mathcal{P}_n(\mathcal{X})$ denotes the set of all types in $\mathcal{X}^n$, and define the set of all sequences in $\mathcal{X}^n$ of type P as

$$T_P \triangleq \{\mathbf{x} \in \mathcal{X}^n : P_{\mathbf{x}} = P\}. \tag{2}$$

The joint type of a pair $(\mathbf{x}, \mathbf{y}) \in \mathcal{X}^n \times \mathcal{Y}^n$ is the probability distribution $P_{\mathbf{x},\mathbf{y}}$ on $\mathcal{X} \times \mathcal{Y}$ defined by:

$$P_{\mathbf{x},\mathbf{y}}(x, y) \triangleq \frac{1}{n} N(x, y|\mathbf{x}, \mathbf{y}), \qquad (x, y) \in \mathcal{X} \times \mathcal{Y}, \tag{3}$$

where $N(x, y|\mathbf{x}, \mathbf{y})$ is the number of occurrences of (x, y) in $(\mathbf{x}, \mathbf{y})$.

Definition 1. *An (n, M, N) multi-user code is a set $\{(\mathbf{x}_i, \mathbf{y}_j, D_{ij}) : 1 \le i \le M, 1 \le j \le N\}$ with*

- $\mathbf{x}_i \in \mathcal{X}^n$, $\mathbf{y}_j \in \mathcal{Y}^n$, $D_{ij} \subset \mathcal{Z}^n$
- $D_{ij} \cap D_{i'j'} = \varnothing$ *for* $(i, j) \neq (i', j')$.

The average error probability of this code for the MAC, $W : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$, is defined as

$$e(\mathcal{C}, W) \triangleq \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} W^n(D_{i,j}|\mathbf{x}_i, \mathbf{y}_j). \tag{4}$$

Similarly, the maximal error probability of this code for W is defined as

$$e_m(\mathcal{C}, W) \triangleq \max_{(i,j)} W^n(D_{i,j}|\mathbf{x}_i, \mathbf{y}_j). \tag{5}$$

Definition 2. *For the MAC, $W : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$, the average and maximal error reliability functions, at rate pair (R_X, R_Y), are defined as:*

$$E^*_{av}(R_X, R_Y) \triangleq \lim_{n\to\infty} \max_{\mathcal{C}} \frac{1}{n} \log e(\mathcal{C}, W) \tag{6}$$

$$E^*_m(R_X, R_Y) \triangleq \lim_{n\to\infty} \max_{\mathcal{C}} \frac{1}{n} \log e_m(\mathcal{C}, W), \tag{7}$$

where the maximum is over all codes of length n and rate pair (R_X, R_Y).

Definition 3. *A code $\mathcal{C}_X = \{\mathbf{x}_i \in T_{P_X} : \; i = 1, ..., M_X\}$, for some P_X, is called a bad codebook, if*

$$\exists\, (i,j), \quad i \neq j \qquad \mathbf{x}_i = \mathbf{x}_j \tag{8}$$

A codebook which is not bad, is called a good one.

Definition 4. *A multi user code $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y$ is called a good multi user code, if both individual codebooks $\mathcal{C}_X$, $\mathcal{C}_Y$ are good codes.*

Definition 5. *For a good multi user code $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y$, and for a particular type $P_{XY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})$, we define*

$$R(\mathcal{C}, P_{XY}) \triangleq \frac{1}{n} \log |\mathcal{C} \cap T_{P_{XY}}| \tag{9}$$

For a specified channel W, the Bhattacharyya distance between the channel input letter pairs (x, y), and $(\tilde{x}, \tilde{y})$ is defined by

$$d_B\big((x,y),(\tilde{x},\tilde{y})\big) \triangleq -\log\big(\sum_{z \in \mathcal{Z}} \sqrt{W(z|x,y)W(z|\tilde{x},\tilde{y})}\big)$$

In this paper, we assume $d_B\big((x,y),(\tilde{x},\tilde{y})\big) \neq \infty$ for all (x,y) and $(\tilde{x},\tilde{y})$. A channel with this property is called an *indivisible channel*. An indivisible channels for which the matrix $A_{(i,j),(k,l)} = 2^{s d_B((i,j),(k,l))}$ is nonnegative-definite for all $s > 0$ is called a *nonnegative-definite* channel.

For a block channel W^n, the normalized Bhattacharyya distance between two channel input block pairs $(\mathbf{x}, \mathbf{y})$, and $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is given by:

$$d_B\big((\mathbf{x},\mathbf{y}),(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\big) = -\frac{1}{n}\log\big(\sum_{\mathbf{z} \in \mathcal{Z}^n} \sqrt{W(\mathbf{z}|\mathbf{x},\mathbf{y})W(\mathbf{z}|\tilde{\mathbf{x}},\tilde{\mathbf{y}})}\big)$$

If W is a memoryless channel, it can be easily shown that the Bhattacharyya distance between two pairs of codewords $(\mathbf{x}, \mathbf{y})$ and $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$, with joint empirical density $P_{XY\tilde{X}\tilde{Y}}$, is

$$d_B\big((\mathbf{x},\mathbf{y}),(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\big) = \sum_{\substack{x,\tilde{x} \in \mathcal{X} \\ y,\tilde{y} \in \mathcal{Y}}} P_{XY\tilde{X}\tilde{Y}}(x,y,\tilde{x},\tilde{y}) d_B\big((x,y),(\tilde{x},\tilde{y})\big)$$

As we see here, for a fixed channel, the Bhattacharyya distance between two pairs of words depends only on their joint composition. The minimum Bhattacharyya distance for a code $\mathcal{C}$ is defined as:

$$d_B(\mathcal{C}) \triangleq \min_{\substack{(\mathbf{x},\mathbf{y}),(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) \in \mathcal{C} \\ (\mathbf{x},\mathbf{y}) \neq (\tilde{\mathbf{x}},\tilde{\mathbf{y}})}} d_B\big((\mathbf{x},\mathbf{y}),(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\big). \tag{10}$$

Let us define

$$d_B^*\big(R_X, R_Y, n\big) \triangleq \max_{\mathcal{C}} d_B(\mathcal{C}) \tag{11}$$

Where the maximum is over all good codes of rate (R_X, R_Y), and blocklength n. Finally, we define

$$d_B^*(R_X, R_Y) \triangleq \lim_{n \to \infty} d_B^*\big(R_X, R_Y, n\big) \tag{12}$$

Note that, since any bad code has at least two identical codewords, we can conclude that the minimum distance of the code is equal to zero. Therefore, in order to find an upper bound for the best possible minimum distance, $d_B^*(R_X, R_Y)$, we only need to consider good codes (codes without any repetitions).

Now, consider any joint type $P_{XY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})$. Using the structure of Bhattacharyya distance function, we can define spheres in $T_{P_{XY}}$. For any $(\mathbf{x}, \mathbf{y}) \in T_{P_{XY}}$, the sphere about $(\mathbf{x}, \mathbf{y})$, of radius r, is given by

$$S \triangleq \{(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) : d_B\big((\mathbf{x}, \mathbf{y}), (\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big) \leq r\}$$

Every point, $(\mathbf{x}, \mathbf{y}) \in T_{P_{XY}}$, is surrounded by a set consisting of all pairs with which it shares some given joint type $V_{XY\tilde{X}\tilde{Y}}$. Basically, any pair of sequences, $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \in T_{P_{XY}}$, sharing a common joint type with some given pair of sequences, $(\mathbf{x}, \mathbf{y}) \in T_{P_{XY}}$, belongs to the surface of a sphere with center $(\mathbf{x}, \mathbf{y})$ and radius $r = d_B\big((\mathbf{x}, \mathbf{y}), (\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big)$. The set of these pairs is called a spherical collection about $(\mathbf{x}, \mathbf{y})$ defined by $P_{\mathbf{x},\mathbf{y},\tilde{\mathbf{x}},\tilde{\mathbf{y}}}$.

III. Minimum Distance Upper Bound

Suppose the number of messages of the first source is $M_X = 2^{nR_X}$, and the number of messages of the second source is $M_Y = 2^{nR_Y}$. Suppose all the messages of any source are equiprobable and the sources are sending data independently. With these assumptions, all $M_X M_Y$ pairs are occuring with the same probability. Thus, at the input of the channel, we can see all possible $2^{n(R_X+R_Y)}$ (an exponentially increasing function of n) pairs of input sequences. However, we also know that the number of possible types is a polynomial function of n. Thus, for at least one joint type, the number of pairs of sequences in the multi user code which have that particular type, should be an exponential function of n with the rate arbitrary close to the rate of the multi user code. We will look at these pairs of sequences as a subcode, and then try to find an upper bound for the minimum distance of this subcode. Clearly, this bound is still a valid upper bound for the minimum distance of the original multi user code.

Lemma 1. *For any $\delta > 0$, and for any good multi user code $\mathcal{C}$ with rate pair (R_X,R_Y), as defined above, there exists $P_{XY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})$ such that*

$$R(\mathcal{C}, P_{XY}) \geq R_X + R_Y - \delta \qquad \textit{for sufficiently large } n$$

Proof: The code $\mathcal{C}$ can be written as

$$\mathcal{C} = \bigcup_{P_{XY}} (\mathcal{C} \cap T_{P_{XY}}) = \bigcup_{P_{XY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})} (\mathcal{C} \cap T_{P_{XY}}) \tag{13}$$

However, for different P_{XY}s, $T_{P_{XY}}$ are disjoint sets. So,

$$|\mathcal{C}| = \sum_{P_{XY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y}):} |\mathcal{C} \cap T_{P_{XY}}| \Rightarrow \tag{14}$$

$$2^{n(R_X+R_Y)} = \sum_{P_{XY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})} 2^{nR(\mathcal{C}, P_{XY})} \tag{15}$$

On the other hand,$|\mathcal{P}_n(\mathcal{X} \times \mathcal{Y})| \leq (n+1)^{|\mathcal{X}||\mathcal{Y}|}$. So,

$$2^{n(R_X+R_Y)} \leq (n+1)^{|\mathcal{X}||\mathcal{Y}|} 2^{n \max_{P_{XY}} R(\mathcal{C}, P_{XY})} \tag{16}$$

Therefore, for sufficiently large n, it is obvious that

$$\max_{P_{XY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})} R(\mathcal{C}, P_{XY}) \geq R_X + R_Y - \delta \tag{17}$$

Thus, there exists at least one $P_{XY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})$ for which the rate of $\mathcal{C} \cap T_{P_{XY}}$ is the same as the rate of the multi user code. ■

Definition 6. *For a sequence of joint types $P^n_{XY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})$, with marginal types P^n_X and P^n_Y, the sequence of type graphs, G_n, is defined as follows. For every n, G_n is a bipartite graph, with its left vertices consisting of all $x^n \in T_{P^n_X}$ and the right vertices consisting of all $y^n \in T_{P^n_Y}$. A vertex on the left (say $\tilde{x}^n$) is connected to a vertex on the right (say $\tilde{y}^n$) if and only if $(\tilde{x}^n, \tilde{y}^n) \in T_{P^n_{XY}}$.*

Lemma 2. *[11] For all sequences of nearly complete subgraphs of a particular type graph $T_{P_{XY}}$, the rates of the subgraph (R_X, R_Y) must satisfy*

$$R_X \leq H(X|U),\ R_Y \leq H(Y|U) \tag{18}$$

for some $P_{U|XY}$ such that $X - U - Y$.

Consider any multiuser codebook $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y$ with dominant type P_{XY}. Consider any joint composition $V_{XY\tilde{X}\tilde{Y}}$ with marginal distributions $V_{XY} = V_{\tilde{X}\tilde{Y}} = P_{XY}$. In the following lemma, we find the average number of pairs of codewords in a spherical collection defined by joint type $V_{XY\tilde{X}\tilde{Y}}$ about an arbitrary pair of sequences $(\mathbf{x}, \mathbf{y}) \in T_{P_{XY}}$. For such $(\mathbf{x}, \mathbf{y})$, which is not necessarily a pair of codewords, let us define the following sets:

- $A_X(\mathbf{x}, \mathbf{y}) \triangleq \{(\mathbf{x}, \tilde{\mathbf{y}}) \in \mathcal{C} : (\mathbf{x}, \mathbf{y}, \mathbf{x}, \tilde{\mathbf{y}}) \in T_{V_{XY\tilde{X}\tilde{Y}}}\}$
- $A_Y(\mathbf{x}, \mathbf{y}) \triangleq \{(\tilde{\mathbf{x}}, \mathbf{y}) \in \mathcal{C} : (, \mathbf{x}, \mathbf{y}, \tilde{\mathbf{x}}, \mathbf{y}) \in T_{V_{XY\tilde{X}\tilde{Y}}}\}$
- $A_{XY}(\mathbf{x}, \mathbf{y}) \triangleq \{(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \in \mathcal{C} : (\mathbf{x}, \mathbf{y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \in T_{V_{XY\tilde{X}\tilde{Y}}}\}$

Note that, if $\mathbf{x} \notin \mathcal{C}_X$ or $X \neq \tilde{X}$, the first set would be empty. Similarly, if $\mathbf{y} \notin \mathcal{C}_Y$ or $Y \neq \tilde{Y}$, the second one would be an empty set.

Lemma 3. *Consider the multi-user code, $\mathcal{C}$, described above with dominant joint type P_{XY}. Additionally, consider any distribution $V_{XY\tilde{X}\tilde{Y}} \in \mathcal{P}\big((\mathcal{X} \times \mathcal{Y})^2\big)$, satisfying $V_{XY} = V_{\tilde{X}\tilde{Y}} = P_{XY}$. Then, there exists a pair of sequences $(\mathbf{x}, \mathbf{y}) \in T_{P_{XY}}$ such that*

$$|A_{XY}(\mathbf{x}, \mathbf{y})| \geq \exp\{n[R_X + R_Y - I(\tilde{X}\tilde{Y} \wedge XY)]\}. \tag{19}$$

Furthermore, for any distribution $V_{XY\tilde{X}} \in \mathcal{P}\big(\mathcal{X} \times \mathcal{Y} \times \mathcal{X}\big)$ satisfying $V_{XY} = V_{\tilde{X}Y} = P_{XY}$, and any $\mathbf{y} \in \mathcal{C}_Y \cap T_{P_Y}$, there exists some $\mathbf{x} \in T_{P_X}$, such that $(\mathbf{x}, \mathbf{y}) \in T_{P_{XY}}$, and

$$|A_Y(\mathbf{x}, \mathbf{y})| \geq \exp\{n[R_X - I(\tilde{X} \wedge X|Y)]\},. \tag{20}$$

Similarly, for any distribution $V_{XY\tilde{Y}} \in \mathcal{P}\big(\mathcal{X} \times \mathcal{Y} \times \mathcal{Y}\big)$ satisfying $V_{XY} = V_{X\tilde{Y}} = P_{XY}$, and any $\mathbf{x} \in \mathcal{C}_X \cap T_{P_X}$, there exists some $\mathbf{y} \in T_{P_Y}$ such that $(\mathbf{x}, \mathbf{y}) \in T_{P_{XY}}$, and

$$|A_X(\mathbf{x}, \mathbf{y})| \geq \exp\{n[R_Y - I(\tilde{Y} \wedge Y|X)]\}. \tag{21}$$

Proof: For a fixed $V_{XY\tilde{X}\tilde{Y}}$, let us study the spherical collection consisting of all pairs of codewords sharing composition $V_{XY\tilde{X}\tilde{Y}}$ with some arbitrary pairs in $T_{P_{XY}}$. Consider such spherical collection for every pair of sequences. Since each of the codewords pairs shares joint composition $V_{XY\tilde{X}\tilde{Y}}$ with $\exp\{H(\tilde{X}\tilde{Y}|XY)\}$ pairs, it must belong to $\exp\{H(\tilde{X}\tilde{Y}|XY)\}$ different spherical collections. Therefore,

$$\sum_{(\mathbf{x},\mathbf{y})\in T_{P_{XY}}} |A_{XY}(\mathbf{x},\mathbf{y})| = \exp\{n[R_X + R_Y + H(\tilde{X}\tilde{Y}|XY)]\}$$

Hence, by dividing both sides of the previous equality by $|T_{P_{XY}}|$, we conclude that

$$\frac{1}{|T_{P_{XY}}|}\sum_{\substack{(\mathbf{x},\mathbf{y})\\ \in T_{P_{XY}}}} |A_{XY}(\mathbf{x},\mathbf{y})| = 2^{n[R_X+R_Y-I(\tilde{X}\tilde{Y}\wedge XY)]}.$$

Thus, there must exist a pair of sequence, $(\mathbf{x},\mathbf{y}) \in T_{P_{XY}}$, with

$$|A_{XY}(\mathbf{x},\mathbf{y})| \geq \exp\{n[R_X + R_Y - I(\tilde{X}\tilde{Y}\wedge XY)]\}. \tag{22}$$

By a similar argument, we can conclude (20) and (21). ■

Lemma 4. *Fix $\epsilon > 0$. Let W be a nonnegative-definite channel. Let $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y$ be any multi-user code with dominant composition nP_{XY} and rate pair (R_X, R_Y). Consider any distribution $V_{XY\tilde{X}\tilde{Y}} \in \mathcal{P}(\mathcal{X}\times\mathcal{Y}\times\mathcal{X}\times\mathcal{Y})$ satisfying the following constraints:*

- $V_{XY} = V_{\tilde{X}\tilde{Y}} = P_{XY}$
- $I_V(XY \wedge \tilde{X}\tilde{Y}) \leq R_X + R_Y - \epsilon$.

Then, $\mathcal{C}$ has two pairs of codewords, $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ and $(\hat{\mathbf{x}},\hat{\mathbf{y}})$, such that

$$d_B\big((\tilde{\mathbf{x}},\tilde{\mathbf{y}}),(\hat{\mathbf{x}},\hat{\mathbf{y}})\big) \leq (1+\epsilon)Ed_B\big((\tilde{X},\tilde{Y}),(\hat{X},\hat{Y})\big)\big) \tag{23}$$

where the expectation is calculated based on $V_{XY\tilde{X}\tilde{Y}\hat{X}\hat{Y}} \in \mathcal{P}((\mathcal{X}\times\mathcal{Y})^3)$ satisfying

- $V_{XY} = V_{\tilde{X}\tilde{Y}} = V_{\hat{X}\hat{Y}} = P_{XY}$
- $\tilde{X}\tilde{Y} - XY - \hat{X}\hat{Y}$
- $V_{\tilde{X}\tilde{Y}|XY} = V_{\hat{X}\hat{Y}|XY}$
- $I_V(XY \wedge \tilde{X}\tilde{Y}) \leq R_X + R_Y - \epsilon$.

For any $V_{XY\tilde{X}} \in \mathcal{P}(\mathcal{X}\times\mathcal{Y}\times\mathcal{X})$ satisfying the following constraints:

- $V_{XY} = V_{\tilde{X}Y} = P_{XY}$
- $I_V(X \wedge \tilde{X}|Y) \leq R_X - \epsilon$.

$\mathcal{C}$ has two pairs of codewords, $(\tilde{\mathbf{x}},\mathbf{y})$ and $(\hat{\mathbf{x}},\mathbf{y})$, such that

$$d_B\big((\tilde{\mathbf{x}},\mathbf{y}),(\hat{\mathbf{x}},\mathbf{y})\big) \leq (1+\epsilon)Ed_B\big((\tilde{X},Y),(\hat{X},Y)\big)\big) \tag{24}$$

where the expectation is calculated based on $V_{XY\tilde{X}\hat{X}} \in \mathcal{P}(\mathcal{X}\times\mathcal{Y}\times\mathcal{X}\times\mathcal{X})$ satisfying

- $V_{XY} = V_{\tilde{X}Y} = V_{\hat{X}Y} = P_{UXY}$
- $\tilde{X} - XY - \hat{X}$

- $V_{\tilde{X}|XY} = V_{\hat{X}|XY}$
- $I_V(X \wedge \tilde{X}|Y) \leq R_X - \epsilon$.

Similarly, for any $V_{XY\tilde{Y}} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y} \times \mathcal{Y})$ *satisfying the following constraints:*

- $V_{XY} = V_{\tilde{Y}} = P_{XY}$
- $I_V(Y \wedge \tilde{Y}|X) \leq R_Y - \epsilon$.

$\mathcal{C}$ *has two pairs of codewords,* $(\mathbf{x}, \tilde{\mathbf{y}})$ *and* $(\mathbf{x}, \hat{\mathbf{y}})$*, such that*

$$d_B\big((\mathbf{x}, \tilde{\mathbf{y}}), (\mathbf{x}, \hat{\mathbf{y}})\big) \leq (1+\epsilon) E d_B\big((X, \tilde{Y}), (X, \hat{Y})\big) \tag{25}$$

where the expectation is calculated based on $V_{XY\tilde{Y}\hat{Y}} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y})$ *satisfying*

- $V_{XY} = V_{X\tilde{Y}} = V_{X\hat{Y}} = P_{XY}$
- $\tilde{Y} - XY - \hat{Y}$
- $V_{\tilde{Y}|XY} = V_{\hat{Y}|XY}$
- $I_V(Y \wedge \tilde{Y}|X) \leq R_Y - \epsilon$.

Proof: The proof is provided in Appendix. ■

Noting that the minimum distance between codeword pairs in $\mathcal{C}$ is smaller than the minimum distance of any subset of $\mathcal{C}$, we conclude the following result.

Theorem 1. *For any nonnegative-definite channel,* W*, the minimum distance of any multiuser code,* $\mathcal{C}$*, with rate pair* (R_X, R_Y) *satisfies*

$$d_B(\mathcal{C}) \leq E_U(R_X, R_Y, W) \tag{26}$$

where $E_U(R_X, R_Y, W)$ *is defined as*

$$\max_{P_{UXY}} \min_{\beta = X, Y, XY} E_U^{\beta}(R_X, R_Y, W, P_{XYU}) \tag{27}$$

The maximum is over all $P_{UXY} \in \mathcal{P}(\mathcal{U} \times \mathcal{X} \times \mathcal{Y})$ *such that* $X - U - Y$*, and* $R_X \leq H(X|U)$ *and* $R_Y \leq H(Y|U)$*. The functions* $E_U^{\beta}(R_X, R_Y, W, P_{XYU})$ *are defined as follows:*

$$E_U^X(R_X, R_Y, W, P_{XYU}) \triangleq \min_{V_{X\tilde{X}\hat{X}Y} \in \mathcal{V}_X^U} E d_W\big((\hat{X}, Y), (\tilde{X}, Y)\big)$$

$$E_U^Y(R_X, R_Y, W, P_{XYU}) \triangleq \min_{V_{XY\tilde{Y}\hat{Y}} \in \mathcal{V}_Y^U} E d_W\big((X, \hat{Y}), (X, \tilde{Y})\big)$$

$$E_U^{XY}(R_X, R_Y, W, P_{XYU}) \triangleq \min_{\substack{V_{XY\tilde{X}\tilde{Y}\hat{X}\hat{Y}} \\ \in \mathcal{V}_{XY}^U}} E d_W\big((\hat{X}, \hat{Y}), (\tilde{X}, \tilde{Y})\big)$$

where

$$
\begin{aligned}
\mathcal{V}_X^U \triangleq \{ & V_{X\tilde{X}\hat{X}Y} : V_{\tilde{X}Y} = V_{\hat{X}Y} = V_{XY} = P_{XY} \\
& \hat{X} - XY - \tilde{X} \\
& V_{\tilde{X}|XY} = V_{\hat{X}|XY} \\
& I(X \wedge \tilde{X}|Y) = I(X \wedge \hat{X}|Y) \leq R_X \}
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
\mathcal{V}_Y^U \triangleq \{ & V_{XY\tilde{Y}\hat{Y}} : V_{X\tilde{Y}} = V_{X\hat{Y}} = V_{XY} = P_{XY} \\
& \hat{Y} - XY - \tilde{Y} \\
& V_{\tilde{Y}|XY} = V_{\hat{Y}|XY} \\
& I(Y \wedge \tilde{Y}|X) = I(Y \wedge \hat{Y}|X) \leq R_Y \}
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
\mathcal{V}_{XY}^U \triangleq \{ & V_{XY\tilde{X}\tilde{Y}\hat{X}\hat{Y}} : V_{\tilde{X}\tilde{Y}} = V_{\hat{X}\hat{Y}} = V_{XY} = P_{XY} \\
& \hat{X}\hat{Y} - XY - \tilde{X}\tilde{Y} \\
& V_{\tilde{X}\tilde{Y}|XY} = V_{\hat{X}\hat{Y}|XY} \\
& I(XY \wedge \tilde{X}\tilde{Y}) = I(XY \wedge \hat{X}\hat{Y}) \leq R_X + R_Y \}
\end{aligned} \tag{30}
$$

Theorem 2. *For any indivisible channel*

$$E_m^*(R_X, R_Y) \leq d_B^*(R_X, R_Y) \tag{31}$$

where $E_m^*(R_X, R_Y)$ *is the maximal error channel-rate reliability function at rate pair* (R_X, R_Y)*.*

Proof: The proof is very similar to [12]. ■

Therefore, by combining the result of theorem 1 and theorem 2, we can conclude the following result.

Theorem 3. *For any indivisible nonnegative-definite channel,* W*, the maximal error reliability function,* $E_m^*(R_X, R_Y)$*, must satisfy*

$$E_m^*(R_X, R_Y) \leq E_U(R_X, R_Y, W) \tag{32}$$

The following observation will be used in section IV and V to compare the lower bound on the average error reliability function with the upper bounds on the maximal error reliability function at $R_X = R_Y = 0$.

Lemma 5. *If* $\min\{R_X, R_Y\} = 0$*, i.e.,* $R_X = 0$ *or* $R_Y = 0$*,*

$$E_m^*(R_X, R_Y) = E_{av}^*(R_X, R_Y) \tag{33}$$

IV. An Expurgated Lower Bound

Theorem 4. *For every* $\delta > 0$*,* $R_X \geq 0$*,* $R_Y \geq 0$ *, every finite set* $\mathcal{U}$*, every type* $\mathcal{P}_{XYU} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y} \times \mathcal{U})$*,* $X - U - Y$*, satisfying* $H(X|U) \geq R_X$ *and* $H(Y|U) \geq R_Y$*, and* $\mathbf{u} \in T_{P_U}^n$*, there exists a multi-user code*

$$\mathcal{C} = \{(\mathbf{x}_i, \mathbf{y}_j, D_{ij}) : i = 1, ...M_X^*, j = 1, ...M_Y^*\} \tag{34}$$

with $\mathbf{x}_i \in T_{P_{X|U}}(\mathbf{u})$, $\mathbf{y}_j \in T_{P_{Y|U}}(\mathbf{u})$ *for all* i *and* j, $M_X^* \geq 2^{n(R_X-\delta)}$, *and* $M_Y^* \geq 2^{n(R_Y-\delta)}$, *such that for every MAC* $W : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$

$$e(\mathcal{C}, W) \leq 2^{-n[E_L(R_X,R_Y,W,P_{XYU})-\delta]} \tag{35}$$

whenever $n \geq n_1(|\mathcal{Z}|, |\mathcal{X}|, |\mathcal{Y}|, |\mathcal{U}|, \delta)$, *where*

$$E_L(R_X, R_Y, W, P_{XYU}) \triangleq \min_{\beta=X,Y,XY} E_L^{\beta}(R_X, R_Y, W, P_{XYU}) \tag{36}$$

and $E_L^{\beta}(R_X, R_Y, W, P_{XYU})$, $\beta = X, Y, XY$ *are defined respectively by*

$$E_L^X(R_X, R_Y, W, P_{XYU}) \triangleq \min_{V_{UXY\tilde{X}} \in \mathcal{V}_X} Ed_W\big((X,Y),(\tilde{X},Y)\big) + I_V(X \wedge Y|U) + I(\tilde{X} \wedge X|YU) + I_V(\tilde{X} \wedge Y|U) - R_X \tag{37}$$

$$E_L^Y(R_X, R_Y, W, P_{XYU}) \triangleq \min_{V_{UXY\tilde{Y}} \in \mathcal{V}_Y} Ed_W\big((X,Y),(X,\tilde{Y})\big) + I_V(X \wedge Y|U) + I(\tilde{Y} \wedge Y|XU) + I_V(X \wedge \tilde{Y}|U) - R_Y \tag{38}$$

$$E_L^{XY}(R_X, R_Y, W, P_{XYU}) \triangleq \min_{V_{UXY\tilde{X}\tilde{Y}} \in \mathcal{V}_{XY}} Ed_W\big((X,Y),(\tilde{X},\tilde{Y})\big) + I_V(X \wedge Y|U) + I(\tilde{X}\tilde{Y} \wedge XY|U) + I_V(\tilde{X} \wedge \tilde{Y}|U) - R_X - R_Y \tag{39}$$

where

$$
\begin{aligned}
\mathcal{V}_X \triangleq \{V_{UXY\tilde{X}} &: V_{XU} = V_{\tilde{X}U} = P_{XU}, V_{YU} = P_{YU} \\
&I_V(X \wedge Y|U), I_V(\tilde{X} \wedge Y|U) \leq \min\{R_X, R_Y\} + 3\delta \\
&I_V(X \wedge Y|U) + I_V(\tilde{X} \wedge Y|U) + I_V(\tilde{X} \wedge X|UY) \\
&\qquad \leq R_X + \min\{R_X, R_Y\} + 4\delta\}
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
\mathcal{V}_Y \triangleq \{V_{UXY\tilde{Y}} &: V_{XU} = P_{XU}, V_{YU} = V_{\tilde{Y}U} = P_{YU} \\
&I_V(X \wedge Y|U), I_V(X \wedge \tilde{Y}|U) \leq \min\{R_X, R_Y\} + 3\delta \\
&I_V(X \wedge Y|U) + I_V(X \wedge \tilde{Y}|U) + I_V(\tilde{Y} \wedge Y|UX) \\
&\qquad \leq R_Y + \min\{R_X, R_Y\} + 4\delta\}
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
\mathcal{V}_{XY} \triangleq \big\{V_{UXY\tilde{X}\tilde{Y}} &: \\
&V_{UXY\tilde{X}} \textit{ satisfies all conditions in } (40) \\
&V_{UXY\tilde{Y}} \textit{ satisfies all conditions in } (41) \\
&I_V(X \wedge \tilde{Y}|U) + I_V(\tilde{X} \wedge \tilde{Y}|U) + I_V(\tilde{X} \wedge X|U\tilde{Y}) \\
&\qquad \leq R_X + \min\{R_X, R_Y\} + 4\delta \\
&I_V(\tilde{X} \wedge Y|U) + I_V(\tilde{X} \wedge \tilde{Y}|U) + I_V(\tilde{Y} \wedge Y|U\tilde{X}) \\
&\qquad \leq R_Y + \min\{R_X, R_Y\} + 4\delta \\
&I_V(X \wedge Y|U) + I_V(\tilde{X} \wedge \tilde{Y}|U) + I_V(\tilde{X}\tilde{Y} \wedge XY|U) \\
&\qquad \leq R_X + R_Y + \min\{R_X, R_Y\big\} + 5\delta \\
&I_V(\tilde{X} \wedge Y|U) + I_V(X \wedge \tilde{Y}|U) + I_V(X\tilde{Y} \wedge \tilde{X}Y|U) \\
&\qquad \leq R_X + R_Y + \min\{R_X, R_Y\} + 4\delta\}
\end{aligned} \tag{42}
$$

Proof:

$$
\begin{aligned}
e_{(i,j)} \leq \sum_{\substack{(k,l) \\ \neq (i,j)}} \sum_{\mathbf{z}: W^n(\mathbf{z}|\mathbf{x}_k,\mathbf{y}_l) \geq W^n(\mathbf{z}|\mathbf{x}_i,\mathbf{y}_j)} & W^n\big(\mathbf{z}|\mathbf{x}_i, \mathbf{y}_j\big) \\
= \sum_{k \neq i} \sum_{\mathbf{z}: W^n(\mathbf{z}|\mathbf{x}_k,\mathbf{y}_j) \geq W^n(\mathbf{z}|\mathbf{x}_i,\mathbf{y}_j)} & W^n\big(\mathbf{z}|\mathbf{x}_i, \mathbf{y}_j\big) \\
+ \sum_{l \neq j} \sum_{\mathbf{z}: W^n(\mathbf{z}|\mathbf{x}_i,\mathbf{y}_l) \geq W^n(\mathbf{z}|\mathbf{x}_i,\mathbf{y}_j)} & W^n\big(\mathbf{z}|\mathbf{x}_i, \mathbf{y}_j\big) \\
+ \sum_{\substack{k \neq i \\ l \neq j}} \sum_{\mathbf{z}: W^n(\mathbf{z}|\mathbf{x}_k,\mathbf{y}_l) \geq W^n(\mathbf{z}|\mathbf{x}_i,\mathbf{y}_j)} & W^n\big(\mathbf{z}|\mathbf{x}_i, \mathbf{y}_j\big),
\end{aligned} \tag{43}
$$

therefore, $e_{(i,j)}$ can be upper-bounded by

$$e_{(i,j)} \leq \sum_{k\neq i}\sum_{\mathbf{z}}\sqrt{W^n(\mathbf{z}|\mathbf{x}_k,\mathbf{y}_j)W^n(\mathbf{z}|\mathbf{x}_i,\mathbf{y}_j)} + \sum_{l\neq j}\sum_{\mathbf{z}}\sqrt{W^n(\mathbf{z}|\mathbf{x}_i,\mathbf{y}_l)W^n(\mathbf{z}|\mathbf{x}_i,\mathbf{y}_j)} + \sum_{\substack{k\neq i\\ l\neq j}}\sum_{\mathbf{z}}\sqrt{W^n(\mathbf{z}|\mathbf{x}_k,\mathbf{y}_l)W^n(\mathbf{z}|\mathbf{x}_i,\mathbf{y}_j)}. \tag{44}$$

In (44), the first inner sum is equal to $\exp\{-nEd_W\big((X,Y),(\tilde{X},Y)\big)\}$ for RV's X, Y, and $\tilde{X}$ of joint distribution $P_{\mathbf{x}_i,\mathbf{x}_k,\mathbf{y}_j}$. Similarly, the second inner sum is equal to $\exp\{-nEd_W\big((X,Y),(X,\tilde{Y})\big)\}$ for RV's X, Y, and $\tilde{Y}$ of joint distribution $P_{\mathbf{x}_i,\mathbf{y}_j,\mathbf{y}_l}$, and the third inner sum is equal to $\exp\{-nEd_W\big((X,Y),(\tilde{X},\tilde{Y})\big)\}$ for RV's X, Y, $\tilde{X}$, and $\tilde{Y}$ of joint distribution $P_{\mathbf{x}_i,\mathbf{y}_j,\mathbf{x}_k,\mathbf{y}_l}$. Using the properties of the code which have been proved in [**?**], the average error probability of the code can be upper bounded by

$$e(\mathcal{C},W) \leq 2^{-n[E_L(R_X,R_Y,W,P_{XYU})-\delta]}. \tag{45}$$

■

Let us focus on the case where both codebooks have rate zero, $R_X = R_Y = 0$. One can easily show that,

$$E_L^X(0,0,P_{XYU}) = Ed_W\big((X,Y),(\tilde{X},Y)\big) \tag{46}$$

$$E_L^Y(0,0,P_{XYU}) = Ed_W\big((X,Y),(X,\tilde{Y})\big) \tag{47}$$

$$E_L^{XY}(0,0,P_{XYU}) = Ed_W\big((X,Y),(\tilde{X},\tilde{Y})\big) \tag{48}$$

where all the expectations in (46)-(48) are calculated based on

$$P_{UXY\tilde{X}\tilde{Y}}(u,x,y,\tilde{x},\tilde{y}) = P_U(u)P_{X|U}(x|u)P_{Y|U}(y|u)P_{X|U}(\tilde{x}|u)P_{Y|U}(\tilde{y}|u). \tag{49}$$

Similarly, at zero rate, E_U^X, E_U^Y, and E_U^{XY} would be equal to

$$E_U^X(0,0,P_{XYU}) = Ed_W\big((\hat{X},Y),(\tilde{X},Y)\big) \tag{50}$$

$$E_U^Y(0,0,P_{XYU}) = Ed_W\big((X,\hat{Y}),(X,\tilde{Y})\big) \tag{51}$$

$$E_U^{XY}(0,0,P_{XYU}) = Ed_W\big((\hat{X},\hat{Y}),(\tilde{X},\tilde{Y})\big) \tag{52}$$

where all the expectations in (66)- (68) are respectively calculated based on

$$P_{\hat{X}\tilde{X}Y}(\hat{x},\tilde{x},y) = P_{X|Y}(\hat{x}|y)P_{X|Y}(\tilde{x}|y)P_Y(y) \tag{53}$$

$$P_{X\hat{Y}\tilde{Y}}(x,\hat{y},\tilde{y}) = P_X(x)P_{Y|X}(\hat{y}|x)P_{Y|X}(\tilde{y}|x) \tag{54}$$

$$P_{\hat{X}\hat{Y}\tilde{X}\tilde{Y}}(\hat{x},\hat{y},\tilde{x},\tilde{y}) = P_{XY}(\hat{x},\hat{y})P_{XY}(\tilde{x},\tilde{y}). \tag{55}$$

V. A Conjectured Tighter Upper Bound

Conjecture 1. *For all sequences of nearly complete subgraphs of a particular type graph $T_{P_{XY}}$, the rates of the subgraph (R_X, R_Y) satisfy*

$$R_X \leq H(X|U),\ R_Y \leq H(Y|U) \tag{56}$$

for some $P_{U|XY}$ such that $X - U - Y$. Moreover, there exists $\mathbf{u} \in T_{P_U}$ such that the intersection of the fully connected subgraph with $T_{P_{XY|U}}(\mathbf{u})$ has the rate (R_X, R_Y).

Based on the result of previous lemma, and by following a similar argument as we did in lemma 3 and lemma 4, we can conclude the following result:

Theorem 5. *For any nonnegative-definite channel, W, the minimum distance of any multiuser code, $\mathcal{C}$, with rate pair (R_X, R_Y) satisfies*

$$d_B(\mathcal{C}) \leq E_C(R_X, R_Y, W) \tag{57}$$

where $E_C(R_X, R_Y, W)$ is defined as

$$\max_{P_{UXY}} \min_{\beta=X,Y,XY} E_C^{\beta}(R_X, R_Y, W, P_{XYU}) \tag{58}$$

The maximum is taken over all $P_{UXY} \in \mathcal{P}(\mathcal{U} \times \mathcal{X} \times \mathcal{Y})$ such that $X - U - Y$, and $R_X \leq H(X|U)$ and $R_Y \leq H(Y|U)$. The functions $E_C^{\beta}(R_X, R_Y, W, P_{XYU})$ are defined as follows:

$$E_C^X(R_X, R_Y, W, P_{XYU}) \triangleq \min_{V_{UX\tilde{X}\hat{X}Y} \in \mathcal{V}_X^C} Ed_W\big((\hat{X}, Y), (\tilde{X}, Y)\big)$$

$$E_C^Y(R_X, R_Y, W, P_{XYU}) \triangleq \min_{V_{UXY\tilde{Y}\hat{Y}} \in \mathcal{V}_Y^C} Ed_W\big((X, \hat{Y}), (X, \tilde{Y})\big)$$

$$E_C^{XY}(R_X, R_Y, W, P_{XYU}) \triangleq \min_{\substack{V_{UXY\tilde{X}\tilde{Y}\hat{X}\hat{Y}} \\ \in \mathcal{V}_{XY}^C}} Ed_W\big((\hat{X}, \hat{Y}), (\tilde{X}, \tilde{Y})\big)$$

where

$$
\begin{aligned}
\mathcal{V}_X^C \triangleq \big\{ & V_{UX\tilde{X}\hat{X}Y} : V_{U\tilde{X}Y} = V_{U\hat{X}Y} = V_{UXY} = P_{UXY} \\
& \hat{X} - UXY - \tilde{X} \\
& V_{\tilde{X}|XYU} = V_{\hat{X}|XYU} \\
& I(X \wedge \tilde{X}|YU) = I(X \wedge \hat{X}|YU) \leq R_X \big\}
\end{aligned} \tag{59}
$$

$$
\begin{aligned}
\mathcal{V}_Y^C \triangleq \big\{ & V_{UXY\tilde{Y}\hat{Y}} : V_{UX\tilde{Y}} = V_{UX\hat{Y}} = V_{UXY} = P_{UXY} \\
& \hat{Y} - UXY - \tilde{Y} \\
& V_{\tilde{Y}|XYU} = V_{\hat{Y}|XYU} \\
& I(Y \wedge \tilde{Y}|UX) = I(Y \wedge \hat{Y}|UX) \leq R_Y \big\}
\end{aligned} \tag{60}
$$

$$
\begin{aligned}
\mathcal{V}_{XY}^C \triangleq \big\{ & V_{UXY\tilde{X}\tilde{Y}\hat{X}\hat{Y}} : V_{U\tilde{X}\tilde{Y}} = V_{U\hat{X}\hat{Y}} = V_{UXY} = P_{UXY} \\
& \hat{X}\hat{Y} - UXY - \tilde{X}\tilde{Y} \\
& V_{\tilde{X}\tilde{Y}|UXY} = V_{\hat{X}\hat{Y}|UXY} \\
& I(XY \wedge \tilde{X}\tilde{Y}|U) = I(XY \wedge \hat{X}\hat{Y}|U) \leq R_X + R_Y \big\}
\end{aligned} \tag{61}
$$

Again, let us focus on the case where both codebooks have rate zero, $R_X = R_Y = 0$. Any $V_{UX\tilde{X}\hat{X}Y} \in \mathcal{V}_X^C$ satisfies the following:

$$X - UY - \tilde{X}, \quad X - UY - \hat{X}, \tag{62}$$

therefore, any $V_{UX\tilde{X}\hat{X}Y} \in \mathcal{V}_X^C$ can be written as

$$P_{X|U} P_{X|U} P_{X|U} P_{Y|U} P_U. \tag{63}$$

Similarly, any $V_{UXY\tilde{Y}\hat{Y}} \in \mathcal{V}_Y^C$ can be written as

$$P_{X|U} P_{Y|U} P_{Y|U} P_{Y|U} P_U, \tag{64}$$

and any $V_{UXY\tilde{X}\tilde{Y}\hat{X}\hat{Y}} \in \mathcal{V}_{XY}^C$ can be written as

$$P_{X|U} P_{Y|U} P_{X|U} P_{Y|U} P_{X|U} P_{Y|U} P_U, \tag{65}$$

Therefore, E_C^X, E_C^Y, and E_C^{XY} would be equal to

$$E_C^X(0, 0, P_{XYU}) = E d_W\big((\hat{X}, Y), (\tilde{X}, Y)\big) \tag{66}$$

$$E_C^Y(0, 0, P_{XYU}) = E d_W\big((X, \hat{Y}), (X, \tilde{Y})\big) \tag{67}$$

$$E_C^{XY}(0, 0, P_{XYU}) = E d_W\big((\hat{X}, \hat{Y}), (\tilde{X}, \tilde{Y})\big) \tag{68}$$

where all the expectations in (66)- (68) are respectively calculated based on

$$P_{U\hat{X}\tilde{X}Y}(u,\hat{x},\tilde{x},y) = P_U(u)P_{X|U}(\hat{x}|u)P_{X|U}(\tilde{x}|u)P_{Y|U}(y|u)$$

$$P_{UX\hat{Y}\tilde{Y}}(u,x,\hat{y},\tilde{y}) = P_U(u)P_{X|U}(x|u)P_{Y|U}(\hat{y}|u)P_{Y|U}(\tilde{y}|u)$$

$$P_{U\hat{X}\hat{Y}\tilde{X}\tilde{Y}}(u,\hat{x},\hat{y},\tilde{x},\tilde{y}) = P_U(u)P_{X|U}(\hat{x}|u)P_{Y|U}(\hat{y}|u)P_{X|U}(\tilde{x}|u)P_{Y|U}(\tilde{y}|u) \tag{69}$$

By comparing E_C^β and E_L^β, we conclude the following theorem

Theorem 6. *At rate $R_X = R_Y = 0$,*

$$E_C^\beta(0,0,P_{XYU}) = E_L^\beta(0,0,P_{XYU}) \tag{70}$$

for $\beta = X, Y, XY$, and therefore $E_C = E_L$.

VI. APPENDIX

Consider the joint type $V_{XY\tilde{X}\tilde{Y}}$ for which we have the following properties

- $V_{XY} = V_{\tilde{X}\tilde{Y}} = P_{XY}$.
- $I(\tilde{X}\tilde{Y} \wedge XY) \le R_X + R_Y - \delta$.

For the moment, let us assume that $X \neq \tilde{X}$ and $Y \neq \tilde{Y}$. Let us choose $(\mathbf{x},\mathbf{y}) \in T_{P_{XY}}$ whose existence is asserted in the previous lemma. Let us call the spherical collection about $(\mathbf{x},\mathbf{y}) \in T_{P_{XY}}$, which is defined by $V_{XY\tilde{X}\tilde{Y}}$, as $\mathbf{S}_{XY}$. Also, call the cardinality of this set by T_{XY}, i.e. $|\mathbf{S}_{XY}| = T_{XY}$. From this point, we are going to study the distance structure of the pairs of codewords that lie in $\mathbf{S}_{XY}$. Since we have so many codewords in this spherical collection, they cannot be far from one another. First, we calculate the average distance between any two pairs in this spherical collection. The average distance is given by

$$d_{av}^{XY} = \frac{1}{T(T-1)} d_{tot}$$

where d_{tot} is obtained by adding up all unordered distances between any two not necessarily distinct pairs of codewords in $\mathbf{S}_{XY}$. In the other words, d_{tot} is defined as

$$d_{tot} = \sum_{(\hat{\mathbf{x}},\hat{\mathbf{y}})\in\mathbf{S}_{XY}} \sum_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\in\mathbf{S}_{XY}} d_B\big((\hat{\mathbf{x}},\hat{\mathbf{y}}),(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\big)$$

where $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ and $(\hat{\mathbf{x}},\hat{\mathbf{y}})$ are not necessarily distinct pairs. Therefore,

$$d_{av} = \frac{1}{n}\frac{1}{T_{XY}(T_{XY}-1)} \sum_{\substack{(\hat{\mathbf{x}},\hat{\mathbf{y}}),\\(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\in\mathbf{S}_{XY}}} \sum_{\substack{i,k\in\mathcal{X}\\j,l\in\mathcal{Y}}} n_{\hat{\mathbf{x}}\hat{\mathbf{y}}\tilde{\mathbf{x}}\tilde{\mathbf{y}}}(i,j,k,l) . d_B\big((i,j),(k,l)\big)$$

where $n_{\hat{\mathbf{x}}\hat{\mathbf{y}}\tilde{\mathbf{x}}\tilde{\mathbf{y}}}(i,j,k,l) \triangleq nP_{\hat{\mathbf{x}}\hat{\mathbf{y}}\tilde{\mathbf{x}}\tilde{\mathbf{y}}}(i,j,k,l)$, and $P_{\hat{\mathbf{x}}\hat{\mathbf{y}}\tilde{\mathbf{x}}\tilde{\mathbf{y}}}$ is the joint composition of $(\hat{\mathbf{x}},\hat{\mathbf{y}})$ and $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$. Furthermore, define the variable $n_{\hat{\mathbf{x}}\hat{\mathbf{y}}\tilde{\mathbf{x}}\tilde{\mathbf{y}}}(i,j,k,l|p)$ as follows:

$$n_{\hat{\mathbf{x}}\hat{\mathbf{y}}\tilde{\mathbf{x}}\tilde{\mathbf{y}}}(i,j,k,l|p) = \begin{cases} 1 & \text{if } \hat{\mathbf{x}}_p = i, \hat{\mathbf{y}}_p = j, \tilde{\mathbf{x}}_p = k, \tilde{\mathbf{y}}_p = l \\ 0 & \text{otherwise} \end{cases}$$

Hence, the average distance can be written as

$$d_{av} = \frac{1}{n}\frac{1}{T_{XY}(T_{XY}-1)}\sum_p \sum_{\substack{(\hat{\mathbf{x}},\hat{\mathbf{y}}),\\ (\tilde{\mathbf{x}},\tilde{\mathbf{y}})\in \mathbf{S}_{XY}}} \sum_{\substack{i,k\in\mathcal{X}\\ j,l\in\mathcal{Y}}} n_{\hat{\mathbf{x}}\hat{\mathbf{y}}\tilde{\mathbf{x}}\tilde{\mathbf{y}}}(i,j,k,l|p) . d_B\big((i,j),(k,l)\big)$$

Let $T_{(i,j)|p}$ be the number of $(\mathbf{x},\mathbf{y}) \in \mathbf{S}_{XY}$ with $\mathbf{x}_p = i$, and $\mathbf{y}_p = j$. Following, the previous can be written as

$$d_{av}^{XY} = \frac{1}{n}\frac{T_{XY}}{T_{XY}-1}\sum_p \sum_{\substack{i,k\in\mathcal{X}\\ j,l\in\mathcal{Y}}} \frac{T_{(i,j)|p}T_{(k,l)|p}}{T_{XY}^2} d_B\big((i,j),(k,l)\big)$$

Moreover, Let us define $\lambda_{(i,j)|p}$ as the fraction of the pairs in $\mathbf{S}_{XY}$ with an (i,j) in their pth component, i.e.,

$$\lambda_{(i,j)|p} \triangleq \frac{T_{(i,j)|p}}{T}. \tag{71}$$

Therefore, d_{av}^{XY} can be written as

$$d_{av}^{XY} = \frac{1}{n}\frac{T_{XY}}{T_{XY}-1}\sum_p \sum_{\substack{i,k\in\mathcal{X}\\ j,l\in\mathcal{Y}}} \lambda_{(i,j)|p}\lambda_{(k,l)|p} d_B\big((i,j),(k,l)\big). \tag{72}$$

In general, λ is an unknown function. However, it must satisfy the following equality

$$\sum_{i\in\mathcal{X},j\in\mathcal{Y}} \lambda_{(i,j)|p} = 1 \qquad \text{for all } p. \tag{73}$$

For the center of the sphere, $(\mathbf{x},\mathbf{y})$, we define $\gamma_{(i,j)|p}$ as

$$\gamma_{(i,j)|p} = \begin{cases} 1 & \text{if } \mathbf{x}_p = i,\ \mathbf{y}_p = j \\ 0 & \text{otherwise} \end{cases}.$$

On the other hand, a valid λ must satisfy the following constraint:

$$\sum_p \lambda_{(i,j)|p}\gamma_{(u,k,l)|p} = n_{XY\tilde{X}\tilde{Y}}(k,l,i,j) \tag{74}$$

for all $i,k \in \mathcal{X}$ and all $j,l \in \mathcal{Y}$. Therefore, we can upper bound d_{av} with

$$d_{av}^{XY} \le \frac{1}{n}\frac{T_{XY}}{T_{XY}-1}\max_{\lambda}\sum_p \sum_{\substack{i,k\in\mathcal{X}\\ j,l\in\mathcal{Y}}} \lambda_{(i,j)|p}\lambda_{(k,l)|p} d_B\big((i,j),(k,l)\big).$$

where the maximization is taken over all λ satisfying (73) and (74). In the following lemma, we will find the maximum.

Lemma 6. *Suppose that W is a nonnegative-definite channel. The average distance between the T_{XY} pairs of codewords in the spherical collection, defined by joint composition $V_{XY\tilde{X}\tilde{Y}}$, satisfies*

$$d_{av}^{XY} \leq \frac{T_{XY}}{T_{XY}-1} \sum_{\substack{i,k\in\mathcal{X}\\ j,l\in\mathcal{Y}}} \sum_{\substack{r\in\mathcal{X},\\ s\in\mathcal{Y}}} f_{XY}\big((i,j),(k,l),(r,s)\big). \tag{75}$$

where $f_{XY}\big((i,j),(k,l),(r,s)\big)$ is defined as

$$\frac{n_{XY\tilde{X}\tilde{Y}}(r,s,i,j)n_{XY\tilde{X}\tilde{Y}}(r,s,k,l)}{n.n_{XY}(r,s)} d_B\big((i,j),(k,l)\big).$$

Proof: Let

$$\lambda^*_{(i,j)|p} = \sum_{k\in\mathcal{X},l\in\mathcal{Y}} \frac{n_{XY\tilde{X}\tilde{Y}}(k,l,i,j)}{n_{XY}(k,l)} \gamma_{(k,l)|p} \tag{76}$$

We are going to prove that λ^* achieves the maximum. It is easy to clarify that λ^* satisfies (73) and (74). Moreover, for all λ satisfying (73) and (74),

$$\begin{aligned}&\sum_p \lambda^*_{(i,j)|p}\lambda_{(k,l)|p}\\ &= \sum_{r\in\mathcal{X},s\in\mathcal{Y}} \frac{n_{XY\tilde{X}\tilde{Y}}(r,s,i,j)}{n_{XY}(r,s)} \sum_p \gamma_{(r,s)|p}\lambda_{(k,l)|p}\\ &= \sum_{r\in\mathcal{X},s\in\mathcal{Y}} \frac{n_{XY\tilde{X}\tilde{Y}}(r,s,i,j)n_{XY\tilde{X}\tilde{Y}}(r,s,k,l)}{n.n_{XY}(r,s)}\end{aligned} \tag{77}$$

By assuming that the channel is nonnegative definite, and by using a similar argument as [**?**, Lemma 6], we can show that λ^* achieves the maximum. Substituting this value for λ completes the proof. ■

Now, let us fix a joint type $V_{XY\tilde{X}} \in \mathcal{P}_n(\mathcal{X}\times\mathcal{Y}\times\mathcal{X})$ for which we have the following properties

- $V_{XY} = V_{\tilde{X}Y} = P_{XY}$
- $I(\tilde{X}\wedge X|Y) \leq R_X - \delta$

Let's choose any $\mathbf{y} \in \mathcal{C}_Y \cap T_{P_Y}$. By Lemma 3, there exist some $\mathbf{x} \in T_{P_X}$ such that $(\mathbf{x},\mathbf{y}) \in T_{P_{XY}}$, and the spherical collection about $(\mathbf{x},\mathbf{y})$ defined by $V_{XY\tilde{X}}$ has many pairs of codewords. Let's call such a sphere as $\mathbf{S}_Y$. Assume that $|\mathbf{S}_Y| = T_Y$. We denote the average distance between any two pairs of codeword belonging to this spherical collection by d_{av}^Y. By doing a similar argument as we did before, we can find an upper bound on the d_{av}^Y. It can be easily shown that

$$d_{av}^Y \leq \frac{T_Y}{T_Y-1} \sum_{i,k\in\mathcal{X}} \sum_{r\in\mathcal{X},s\in\mathcal{Y}} f_Y\big(i,k,(r,s)\big). \tag{78}$$

where $f_Y\big(i,k,(r,s)\big)$ is defined as

$$\frac{n_{XY\tilde{X}}(r,s,i)n_{XY\tilde{X}}(r,s,k)}{n.n_{XY}(r,s)} d_B\big((i,j),(k,j)\big).$$

Similarly, let's fix a joint type $V_{XY\tilde{Y}} \in \mathcal{P}_n(\mathcal{X}\times\mathcal{Y}\times\mathcal{Y})$ for which we have the following properties

- $V_{XY} = V_{X\tilde{Y}} = P_{XY}$
- $I(\tilde{Y}\wedge Y|X) \leq R_Y - \delta$

Choose any $\mathbf{x} \in \mathcal{C}_X \cap T_{P_X}$. By Lemma 3, there exist some $\mathbf{y} \in T_{P_Y}$ such that $(\mathbf{x}, \mathbf{y}) \in T_{P_{XY}}$, and the spherical collection about $(\mathbf{x}, \mathbf{y})$ defined by $V_{XY\tilde{Y}}$ has many pairs of codewords. Let's call such a sphere as $\mathbf{S}_X$. Assume that $|\mathbf{S}_X| = T_X$. We denote the average distance between any two pairs of codeword belonging to this spherical collection by d_{av}^X. By doing a similar argument as we did before, we can find an upper bound on the d_{av}^X. It can be easily shown that

$$d_{av}^X \leq \frac{T_X}{T_X - 1} \sum_{j,l \in \mathcal{Y}} \sum_{r \in \mathcal{X}, s \in \mathcal{Y}} f_X\big(j, l, (r,s)\big). \tag{79}$$

where $f_X\big(j, l, (r,s)\big)$ is defined as

$$\frac{n_{XY\tilde{Y}}(r,s,j) n_{XY\tilde{Y}}(r,s,l)}{n.n_{XY}(r,s)} d_B\big((i,j),(i,l)\big).$$

Proof: By using the result of Lemma 3, we can conclude that for any $V_{XY\tilde{X}\tilde{Y}}$ satisfying the mentioned constraints, there exists a pair of sequences $(\mathbf{x}, \mathbf{y}) \in T_{P_{XY}}$, such that the spherical collection about $(\mathbf{x}, \mathbf{y})$ and defined by $V_{XY\tilde{X}\tilde{Y}}$ has exponential many codeword pairs around. Therefore, for sufficiently large n,

$$\frac{T_{XY}}{T_{XY} - 1} \leq 1 + \epsilon \tag{80}$$

Therefor by substituting this upper bound, and by simplifying the result of Lemma 6, we observe that

$$d_{av}^{XY} \leq E d_B\big((\tilde{X}, \tilde{Y}), (\hat{X}, \hat{Y}))\big) \tag{81}$$

The expectation is calculated based on $V_{\tilde{X}\tilde{Y}|XY} V_{\hat{X}\hat{Y}|XY} V_{XY}$. Since the average distance between the pairs in $\mathbf{S}^{XY}$ is greater than some number, there must exist and least two pairs of codewords in $\mathbf{S}^{XY}$ satisfying the same constraints. By a similar argument, we can show the correctness of the second and third part of the theorem. ■

REFERENCES

[1] R. Ahlswede, "Multi-way communication channels," in *Proc. International Symposium on Information Theory*, 1971.

[2] H. Liao, "A coding theorem for multiple-access communications," in *Proc. International Symposium on Information Theory*.

[3] D. Slepian and J. K. Wolf, "A coding theorem for multiple access channels with correlated sources," *bell Syst. tech. J.*, vol. 52, pp. 1037–1076, 1973.

[4] A. G. Dyachkov, "Random constant composition codes for multiple-access channels," *Probl. of Control and Inform. Theory*, pp. 357–369, 1984.

[5] R. Gallager, "A perspective on multi-access channels," *IEEE Trans. Information Theory*, vol. 31, no. 2, pp. 124–142, Mar. 1985.

[6] J. Pokorney and H. S. Wallmeier, "Random coding bounds and codes produced by permutations for the multiple-access channels," *IEEE Trans. Information Theory*, vol. 31, no. 6, pp. 741–750, Nov. 1985.

[7] Y. Liu and B. L. Hughes, "A new universal random coding bound for the multiple-access channels," *IEEE Trans. Information Theory*, vol. 42, no. 2, pp. 376–386, Mar. 1996.

[8] A. Nazari, A. Anastasopoulous, and S. S. Pradhan, "A new universal random-coding bound for average probability error exponent for multiple-access channels," in *Proc. Conference on Information Sciences and Systems (CISS)*, Mar. 2009, online: http://arxiv.org/abs/0901.0948.

[9] E. A. Haroutunian, "Lower bound for the error probability of multiple-access channels," *Problemy Peredachi Informatsii*, vol. 11, pp. 23–36, June 1975.

[10] A. Nazari, S. S. Pradhan, and A. Anastasopoulous, "A new sphere-packing bound for maximal error exponent for multiple-access channels," in *Proc. International Symposium on Information Theory*, 2008, online: http://arxiv.org/abs/0803.3645.

[11] A. Nazari, D. Krithivasan, R. Venkataramanan, S. S. Pradhan, and A. Anastasopoulous, “Typicality graphs and their properties,” in *Proc. International Symposium on Information Theory*, 2009, online: http://arxiv.org/.

[12] R. E. Blahut, “Composition bounds for channel block codes,” *IEEE Trans. Information Theory*, vol. 23, no. 6, pp. 656–674, Nov. 1977.