Characterizing Non-Myopic Information Cascades in Bayesian learning

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Abstract

We consider an environment where many players need to decide whether to buy a certain product (or adopt a trend) or not. The product is either good or bad, but its true value is not known to the players. Instead, each player has his own private information on the quality of the product. Each player can observe the previous actions of other players and deduce the quality of the product. A player can only buy the product once. In contrast to the existing literature on informational cascades, in this work players get more than one opportunity to act. In each turn, a player is chosen uniformly at random from all players and can decide to buy or not to buy. His utility is the total expected discounted reward, and thus myopic strategies may not be best responses. We provide a characterization of structured perfect Bayesian equilibria (PBE) with non-myopic strategies through a fixed-point equation of dimensionality that grows only polynomially with the number of players. Based on this characterization we study informational cascades and show that they happen with high probability for a large number of players. Furthermore, only a small portion of the total information in the system is revealed before a cascade occurs.

I. Introduction

When a new trend or product comes out, one cannot be certain about its quality yet. Many people together might have a better idea about the quality, but in a strategic environment players act individually. Hence, other players' opinions can be revealed only by their actions. This means that waiting to see what other people do might provide more certainty about the quality of the product. On the other hand, many products or trends which turn out to be beneficial are better to be adopted as early as possible, since their value can decay over time.

This interaction can be formalized using a dynamic game with asymmetric information and a discounted utility. Players want to maximize their overall utility, so they might postpone their decision to buy until more information is revealed. This generalizes the sequential Bayesian learning to a setting with non-myopic players and no predefined order of play.

Sequential learning has been extensively explored in the literature, with a special focus on a phenomenon known as an informational cascade [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11]. This is a situation when no player has an incentive to reveal his private information, hence learning stops in the system. This is an interesting case of herd behavior that happens even with fully rational players. In [1], [2], [3], [4], [5], [6], [7], [10], [8], players

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act in a sequence that is predefined before the game starts. When the turn of a certain player arrives, he has no choice but to either buy the product if it seems profitable to him at the moment or forever forgo the opportunity. In this case, it is natural to wonder whether cascades occur because this one-shot opportunity was forced upon the players. Maybe if players had the freedom to choose to wait and gather more information about the product, a herd behavior, especially a wrong one, could have been avoided. This question provides the motivation for studying information cascades in more complex environments. In [12], informational cascades were defined for a general dynamic scenario. However, no evidence for their occurrence was provided.

From a technical perspective, the sequential one-shot framework is conceptually easier since players do not have to account for how much their estimation on the value of the product is going to improve by waiting. This is simply because players are given a single opportunity to act, and cannot wait. This renders the equilibrium analysis of this scenario to be trivial. The equilibrium consists of players that play their best-response in a sequential manner. In the non-myopic scenario, the appropriate solution concept for dynamic games with asymmetric information is a perfect Bayesian Equilibrium (PBE). Finding a PBE is crucial for establishing whether an informational cascade occurs.

Analyzing the PBE of dynamic games with asymmetric information is by itself an interesting field of study. Finding a PBE in a general dynamic scenario with asymmetric information is an extremely challenging task. In [13], [14], [15], the independent types of players was exploited to introduce time-invariant structured strategies, that lead to equations that can be solved for the (structured) PBE. Recently, the case of conditionally independent types [16] and of dependent types [17] were studied and a systematic way to characterize structured PBEs was developed. This characterization is not very useful in analysis because it results in a fixed-point (FP) equation over beliefs, i.e., over an uncountably-infinite space.

The contributions of this paper are the analysis of PBE in a non-myopic scenario and proving that informational cascades are likely to happen. In particular, we characterize structured PBE for the non-myopic scenario with no predefined order of play through a FP equation of finite dimension. We further exploit the structure of the problem to construct even simpler equations with polynomial (quadratic) dimension in the number of players N. These equations can be solved numerically in practice even for relatively large N. We prove that if a solution to those equations exists, then the probability for a cascade approaches one as the number of players approaches infinity. Moreover, we show that the number of players who have revealed their information before the cascade is small - which formalizes their inefficiency. Hence, our work provides a methodology to validate informational cascade in large non-myopic scenarios. This is the first concrete evidence that informational cascades represent a much more general phenomenon that has nothing to do with myopic decisions.

The rest of this paper is organized as follows. In section II we formulate the game of non-myopic players and characterize structured PBEs through a FP equation on appropriate beliefs. In section III we summarize the information contained in the aforementioned beliefs. Based on this summary we provide PBE characterizations through FP equations with finite dimension In section IV we construct further summarized equations with a polynomial dimension in N. In Section V we show that an informational cascade happens with high probability (for large N). Conclusions are drawn in Section VI.

II. Problem Formulation

Define an infinite horizon dynamic game with N players. Time is discrete and the current turn is denoted by t, starting from $t = 0$. At each turn, a player is chosen uniformly at random to act, independently between turns. Only a single player acts each turn. The random index of the acting player at time t is denoted N_t, and its realization is n_t.

There is a product with a random state $V \in \mathcal{V} = \{-1, 1\}$ where $V = -1$ means that the product is bad and $V = 1$ means that the product is good. We define $Q(v) = \Pr(V = v)$ and assume that $Q(v = 1) = Q(v = -1) = \frac{1}{2}$.

Each player has its own private information on the product. The private information of player n is the random variable $X^n \in \mathcal{X} \triangleq \{-1, 1\}$, with distribution

$$Q(x^n|v) \doteq \Pr(X^n = x^n \,|\, V = v) = \begin{cases} 1-p & x^n = v \\ p & x^n \neq v \end{cases} \tag{1}$$

where $p < \frac{1}{2}$. Define the vector of private information as $X = (X_1, ..., X_N)$. The private information is independent between players conditioned on the true value of V, so

$$\Pr(X = x|V = v) = \prod_{n=1}^{N} Q(x^n|v). \tag{2}$$

Player n's action at turn t, denoted by a_t^n, is equal to 1 if player n decides to buy the product at time t and 0 otherwise. Nevertheless, only player n_t can buy the product at time t.

Denote $a_{0:t-1} = (a_0, ..., a_{t-1})$ and $n_{0:t} = (n_0, ..., n_t)$, where $a_t = (a_t^n)_{n \in \mathcal{N}}$. The total history of the game is

$$h_t = (v, x, a_{0:t-1}, n_{0:t}) \in \mathcal{H}_t. \tag{3}$$

We assume each player can observe all the previous actions taken by his opponents. Hence the common history is

$$h_t^c = (a_{0:t-1}, n_{0:t}) \in \mathcal{H}_t^c. \tag{4}$$

These actions provide him with additional information about the quality of the product. Together with his private information, they form the private history of player n

$$h_t^n = (x^n, a_{0:t-1}, n_{0:t}) \in \mathcal{H}_t^n. \tag{5}$$

We define $b_t = (b_t^n)_{n \in \mathcal{N}}$ with b_t^n equal to '1' if and only if player n has bought the product up to time t.

A player's pure strategy is a sequence of functions from the private histories of the game to the action space (i.e., a decision whether to buy or not). In this work, we consider only pure strategies. Formally, player n's strategy is $s^n = (s_t^n)_{t=0}^{\infty}$, with

$$s_t^n : \mathcal{H}_t^n \to \mathcal{A}^n(b_t^n, n_t) \tag{6}$$

where

$$\mathcal{A}^n(b_t^n, n_t) = \begin{cases} \{0, 1\} & b_t^n = 0\,, n_t = n \\ \{0\} & o.w. \end{cases} \tag{7}$$

so that any player n can buy the product only once, and $a_t = 0$ for all t afterwards. In all the turns in which we say that player n does not act, he is compelled not to buy ("play zero"). The instantaneous reward of player n is given by

$$\rho^n(h_t) = \rho(v, a_t^n) = v a_t^n \tag{8}$$

So $\rho(v, a_t^n) = v$ only in the first time that player n buys the product, and 0 in any other case.

Note that for player n, the unknown variables in h_t are X^{-n} and V. Hence, we define the private belief of player n on the history of the game as $\mu_t^n : \mathcal{H}_t^n \to \mathcal{P}(\mathcal{X}^{-n}, \mathcal{V})$. Taking the expectation with respect to this belief and the strategies in (6), we define the expected reward-to-go of player n as

$$R^n(s_{t:\infty}, h_t^n) = \mathbb{E}^{(s,\mu_t^n)} \left\{ \sum_{t'=t}^{\infty} \delta_n^{t'-t} \rho(V, A_{t'}^n) \mid h_t^n \right\} \tag{9}$$

where $\delta_n^\infty = 0$. Note that no more than a single term in the sum (9) can be non-zero.

The strategies in (6) are a function of x^n, $\boldsymbol{a}_{0:t-1}$ and $n_{0:t}$. While $a_{0:t-1}$ and $n_{0:t}$ are observable by all players, x^n is only known to player n. Throughout the paper, it will be useful to decompose those strategies as follows.

Definition II.1. Player n at time t observes h_t^c and takes an action $\gamma_t(x^n)$, where $\gamma_t : \mathcal{X} \to \mathcal{A}^n(b_t^n, n_t)$ is the partial function from his private information to his action. These functions are generated through some policy

$$\psi_t : \mathcal{H}_t^c \to \{\mathcal{X} \to \mathcal{A}^n\} \tag{10}$$

which operates on h_t^c and returns a mapping from x^n to an action, so $\gamma_t = \psi_t[h_t^c]$ and $a_t = \psi_t[h_t^c](x^n)$.

Note that there are only four possible deterministic gamma functions γ_t: wait for any x^n, buy for any x^n, buy according to x^n and buy according to $-x^n$. The last one is clearly dominated by one of the other three so it is never considered. Hence, we remain with three possible functions, i.e., $\gamma \in \{\mathbf{0}, \mathbf{1}, \boldsymbol{I}\}$.

A. Perfect Bayesian Equilibrium

Our main goal is to study when an informational cascade occurs. An informational cascade is defined as a state of the game where learning stops since actions no longer reveal new information. To do so, we first have to study the equilibrium strategies of this game. Since this is a dynamic game with asymmetric information, the appropriate solution concept is a Perfect Bayesian Equilibrium (PBE), defined as follows.

Definition II.2. A PBE with pure strategies is a pair (s^*, μ^*) of

- A strategy profile $s^* = (s^{n*})_{n=1}^N$.
- A belief profile sequence $\mu^* = (\boldsymbol{\mu}_0^*, \boldsymbol{\mu}_1^*, ...)$, where the belief profile at time t is $\boldsymbol{\mu}_t^* = \left(\mu_t^{1*}, ..., \mu_t^{N*}\right)$.

such that sequential rationality holds - for each n, t and $h_t^n \in \mathcal{H}_t^n$, and each s^n

$$R^n\left(s_{t:\infty}^{*n}, s_{t:\infty}^{*-n}, h_t^n\right) \geq R^n\left(s_{t:\infty}^n, s_{t:\infty}^{*-n}, h_t^n\right). \tag{11}$$

B. Characterization of Structured PBE

We now present a methodology for characterizing structured PBEs where the strategy for the acting player n_t depends on the common history only through the common belief on the variables V, X (as well as the variable B_{t-1}). In particular, we define the common belief $\pi_t \in \mathcal{P}\left(\mathcal{V} \times \mathcal{X}^N\right)$ where $\pi_t(x,v) := P\left(X = x, V = v | a_{0:t-1}, b_{0:t-1}, n_{0:t}\right)$. We consider policies of the form $a_t = \psi_t\left[h_t^c\right]\left(x^{n_t}\right) = \theta_t\left[\pi_t\right]\left(x^{n_t}\right)$, i.e., $\gamma_t = \theta_t\left[\pi_t\right]$. In the following we will present a fixed point equation for characterizing the mapping $\theta[.]$ .We first show that the belief π_t can be updated using only public information.

Lemma II.3. *The belief π_t can be updated according to $\pi_{t+1} = F(\pi_t, \gamma_t, a_t, n_t)$. In particular, if $\gamma_t \neq \boldsymbol{I}$, the belief is not updated.*

Proof: By simple application of Bayes' rule we have

$$\pi_{t+1}\left(x,v\right) = \Pr\left(x, v | a_{0:t}, b_{0:t}, n_{0:t+1}\right) \tag{12a}$$

$$= \Pr\left(x, v | a_{0:t}, b_{0:t-1}, n_{0:t}\right) \tag{12b}$$

$$= \frac{\Pr\left(x, v, a_t | a_{0:t-1}, b_{0:t-1}, n_{0:t}\right)}{\Pr\left(a_t | a_{0:t-1}, b_{0:t-1}, n_{0:t}\right)} \tag{12c}$$

$$= \frac{\Pr\left(a_t | x, v, a_{0:t-1}, b_{0:t-1}, n_{0:t}\right) \Pr\left(x, v | a_{0:t-1}, b_{0:t-1}, n_{0:t}\right)}{\Pr\left(a_t | a_{0:t-1}, b_{0:t-1}, n_{0:t}\right)} \tag{12d}$$

$$= \frac{\mathbf{1}_{\gamma_t(x^{n_t})}\left(a_t\right) \pi_t\left(x,v\right)}{\sum_{x',v'} \mathbf{1}_{\gamma_t(x'^{n_t})}\left(a_t\right) \pi_t\left(x',v'\right)} \tag{12e}$$

Whenever the denominator is zero we can set $\pi_{t+1} = \pi_t$. ■

Once the mapping $\theta\left[\cdot\right]$ has been found through the FP equation (13), the sPBE strategies and beliefs are generated through a forward recursion similar to the one in [17], as listed in (14).

Theorem II.4. *Whenever the FP equation* (13) *has a solution, the forward construction described in* (14) *generates a PBE.*

Proof: Due to space limitations we do not prove this result here. The reader is referred to [13], [14], [15] for the details of the proof of a similar (simpler) case with independent types, and to [17] for the proof of a more general case of dependent and time-varying types that subsumes this model[1]. This result can also be found in [16, Chapter 5]. ■

Note that the value functions evaluated in the FP equation (13) are functions of continuous beliefs $\pi \in \mathcal{P}\left(\mathcal{V} \times \mathcal{X}^N\right)$, i.e., on pmf's on a set of size 2^{N+1} which renders this characterization inadequate for analytical and even numerical evaluations due to the infinite dimensions. In the next section we show that due to the structure of the problem, these equations can be simplified considerably. In particular we first show that the domain of these value functions can be reduced to a finite set. We then show that with homogeneous players (i.e., players having the same δ_n), the dimensionality of the fixed point equation is polynomial (quadratic) wrt N.

[1]We note that our results predate the more general treatment in [17].

Fixed-Point Equation

For every $n \in \mathcal{N}$, $\pi \in \mathcal{P}\left(\mathcal{V} \times \mathcal{X}^N\right)$, $b \in \{0,1\}^N$ we evaluate $\gamma^* = \theta\left[n, \pi, b\right]$ as follows

- If $b^n = 1$ then $\gamma^* = \mathbf{0}$.
- If $b^n = 0$ then γ^* is the solution of the following system of equations, $\forall x^n \in \mathcal{X}$

$$\gamma^*\left(x^n\right) = \arg\max\left\{\underbrace{\frac{\delta_n}{N}\sum_{n'=1}^{N} V^n\left(x^n, n', F\left(\pi, \gamma^*, 0, n\right), b\right)}_{0=\text{"don't buy"}}, \underbrace{\sum_v v\pi\left(v|x^n\right)}_{1=\text{"buy"}}\right\} \tag{13a}$$

where the value functions for all $m \in \mathcal{N}$ satisfy

$$V^m\left(x^m, n, \pi, b\right) = \begin{cases} 0, & b^m = 1 \\ \frac{\delta_m}{N}\sum_{n'=1}^{N} V^m\left(x^m, n', F\left(\pi, \gamma^*, 0, m\right), b\right), & b^m = 0, n = m, \gamma^*\left(x^m\right) = 0 \\ \sum_v v\pi\left(v|x^m\right), & b^m = 0, n = m, \gamma^*\left(x^m\right) = 1 \\ \frac{\delta_m}{N}\sum_{n'=1}^{N} \mathbb{E}\left[V^m\left(x^m, n', F\left(\pi, \gamma^*, \gamma^*\left(X^n\right), n\right), b^{-n}B^n\right)\right], & b^m = 0, n \neq m, \end{cases} \tag{13b}$$

where expectation in (13b) is wrt the RVs X^n and B^n with $\Pr\left(X^n = x^n, B^n = b'^n | x^m, n, \pi, b\right) = \Pr\left(B^n = b'^n | X^n = x^n, x^m, n, \pi, b\right)\Pr\left(X^n = x^n | x^m, n, \pi, b\right)$ where

$$\Pr\left(B^n = 1 | X^n = x^n, x^m, n, \pi, b\right) = \begin{cases} 1 & \text{, if } b^n = 1 \text{ or } \gamma^*\left(x^n\right) = 1 \\ 0 & \text{, else,} \end{cases} \tag{13c}$$

and $\Pr(X^n = x^n | x^m, n, \pi, b) = \sum_{\tilde{v}} \pi(x^n|\tilde{v})\pi(\tilde{v}|x^m)$.

Forward Recursion

1) Initialize at time $t = 0$,

$$\mu_0^*\left[\phi\right]\left(v, x\right) := Q(v)\prod_{i=1}^{N} Q(x^i|v). \tag{14a}$$

2) For $t = 0, 1, 2 \ldots$, $\forall n \in \mathcal{N}$, $h_{t+1}^c \in \mathcal{H}_{t+1}^c$, $x^n \in \mathcal{X}$

$$s_t^{n*}\left(h_t^n\right) := \begin{cases} \theta_t\left[n_t, \mu_t^*\left[h_t^c\right], b_{t-1}\right]\left(x^n\right) & n = n_t \\ 0 & o.w. \end{cases} \tag{14b}$$

and

$$\mu_{t+1}^*\left[h_{t+1}^c\right] := F\left(\mu_t^*\left[h_t^c\right], \theta_t\left[n_t, \mu_t^*\left[h_t^c\right], b_{t-1}\right], a_t, n_t\right). \tag{14c}$$

the private beliefs μ_t^{n*} are generated from μ_t^* as

$$\mu_t^{n*}\left(x^{-n}, v\right) = \frac{\mu_t^*\left(x, v\right)}{\mu_t^*\left(x^n\right)} \tag{14d}$$

III. Computing a PBE through a finite-dimensional fixed-point equation

In this section we exploit the structure of the problem to summarize the beliefs. We decompose the belief of each player on the unknown variables x^{-n} and v and show that each part can be updated recursively. Moreover, these parts can be summarized into an integer vector. These two facts together allow to greatly simplify the FP equations on the PBE, such that they become FP equations on $\mathbb{R}^d$ with d finite, and therefore tractable.

Naturally, a player is only interested in the previous actions since they carry information about the product. However, not every action reveals the private information of the acting player. For that to happen, the action that the player took must be determined by his private information.

Definition III.1. Define the revealed information of player n up to time t as the variable $\tilde{x}_t^n \in \{0,-1,1\}$ with the meaning that $\tilde{x}_t^n = 0$ implies the user has not yet revealed his private information, while $\tilde{x}_t^n = \pm 1$ means the player has revealed and the value is as indicated. Note that the quantity $\tilde{x}_t^n$ can be recursively updated as

$$\widetilde{x}_t^n = \begin{cases} 2a_t^n - 1 & \gamma_t = \boldsymbol{I},\, n_t = n, \tilde{x}_{t-1}^n = 0 \\ \widetilde{x}_{t-1}^n & o.w. \end{cases} \tag{15}$$

with the initial condition $\widetilde{x}_0^n = 0$. Note that $\widetilde{x}_t^n$ is a function of $\tilde{x}_{0:t-1}^n$, $a_{0:t}$ and $n_{0:t}$, or equivalently of $\gamma_{0:t}$, $a_{0:t}$ and $n_{0:t}$. We use the notation $\tilde{x}_t = \tilde{F}(\tilde{x}_{t-1}, \gamma_t, a_t, n_t)$ to summarize the recursive update of $\tilde{x}_t$.

The following lemma shows that the common belief decomposes into a belief on v and a belief on x. Specifically, it proves that the private information variables $\{x^m\}$ are conditionally independent given $v\,, h_t^c$.

Lemma III.2. *The belief $\pi_t\,(x,v) = \Pr\,(X = x, V = v|h_t^c)$ can be decomposed as follows*

$$\pi_t\,(x,v) = \pi_t\,(v) \prod_{m=1}^{N} \pi_t\,(x^m|v) \tag{16}$$

where $\pi_t\,(v) \triangleq \Pr\,(V = v\,|\,h_t^c)$ and $\pi_t\,(x^m|v) \triangleq \Pr\,(X^m = x^m\,|\,v, h_t^c)$.

Furthermore,

$$\pi_t\,(x^m|v) = \begin{cases} \boldsymbol{I}_{\tilde{x}_t^m}(x^m) & \widetilde{x}_t^m \neq 0 \\ Q\,(x^m|v) & \widetilde{x}_t^m = 0 \end{cases} \tag{17}$$

and the belief on V can be updated as

$$\frac{\pi_{t+1}(1)}{\pi_{t+1}(-1)} = \frac{\pi_t(1)}{\pi_t(-1)} \times \begin{cases} \left(\frac{1-p}{p}\right)^{2a_t - 1} & \gamma_t = \boldsymbol{I} \text{ and } \tilde{x}_t^{n_t} = 0 \\ 1 & o.w. \end{cases} \tag{18}$$

Finally, the belief on V can be explicitly expressed as

$$\frac{\pi_t(1)}{\pi_t(-1)} = \left(\frac{1-p}{p}\right)^{\sum_n \widetilde{x}_t^n}. \tag{19}$$

Proof: See Appendix A. ■

Using the above structural results for the beliefs, we can simplify the fixed-point equation in (13). The resulting simplified fixed-point equation is shown in (20).

The next theorem concludes this section by showing that (20) is equivalent to (13).

Finite-Dimension Fixed-Point Equation

Denote $q = \frac{1-p}{p}$. For every $n \in \mathcal{N}$, $\tilde{x} \in \{-1,0,1\}^N$, $b \in \{0,1\}^N$ we evaluate $\gamma^* = \theta\left[n, \tilde{x}, b\right]$ as follows

- If $b^n = 1$ then $\gamma^* = \mathbf{0}$.
- If $b^n = 0$ then γ^* is the solution of the following system of equations, $\forall x^n \in \mathcal{X}$

$$\gamma^*\left(x^n\right) = \arg\max \left\{ \underbrace{\frac{\delta_n}{N} \sum_{n'=1}^{N} V^n\left(x^n, n', \tilde{F}\left(\tilde{x}, \gamma^*, 0, n\right), b\right)}_{0=\text{“don't buy”}}, \underbrace{\frac{q^{\sum_m \widetilde{x}^m + x^n \mathbf{1}_0(\widetilde{x}^n)} - 1}{q^{\sum_m \widetilde{x}^m + x^n \mathbf{1}_0(\widetilde{x}^n)} + 1}}_{1=\text{“buy”}} \right\} \tag{20a}$$

where the value functions for all $m \in \mathcal{N}$ satisfy

$$V^m\left(x^m, n, \tilde{x}, b\right) = \begin{cases} 0, & b^m = 1 \\ \frac{\delta_m}{N} \sum_{n'=1}^{N} V^m\left(x^m, n', \tilde{F}\left(\tilde{x}, \gamma^*, 0, m\right), b\right), & b^m = 0, n = m, \gamma^*\left(x^m\right) = 0 \\ \frac{q^{\sum_{m'} \widetilde{x}^{m'} + x^m \mathbf{1}_0(\widetilde{x}^m)} - 1}{q^{\sum_{m'} \widetilde{x}^{m'} + x^m \mathbf{1}_0(\widetilde{x}^m)} + 1}, & b^m = 0, n = m, \gamma^*\left(x^m\right) = 1 \\ \frac{\delta_m}{N} \sum_{n'=1}^{N} \mathbb{E}\left[V^m\left(x^m, n', \tilde{F}\left(\tilde{x}, \gamma^*, \gamma^*\left(X^n\right), n\right), b^{-n} B^n\right)\right], & b^m = 0, n \neq m, \end{cases} \tag{20b}$$

where expectation in (20b) is wrt the RVs X^n and B^n with $\Pr\left(X^n = x^n, B^n = b'^n | x^m, n, \tilde{x}, b\right) = \Pr\left(B^n = b'^n | X^n = x^n, x^m, n, \tilde{x}, b\right) \Pr\left(X^n = x^n | x^m, n, \tilde{x}, b\right)$ where

$$\Pr\left(B^n = 1 | X^n = x^n, x^m, n, \tilde{x}, b\right) = \begin{cases} 1 & \text{, if } b^n = 1 \text{ or } \gamma^*\left(x^n\right) = 1 \\ 0 & \text{, else,} \end{cases} \tag{20c}$$

and

$$\Pr(X^n = x^n | x^m, n, \tilde{x}, b) = \begin{cases} \mathbf{1}_{\tilde{x}^n}\left(x^n\right) & \text{, if } \tilde{x}^n \neq 0 \\ \frac{Q(x^n|-1) + Q(x^n|1) q^{\sum_{m'} \widetilde{x}^{m'} + x^m \mathbf{1}_0(\widetilde{x}^m)}}{1 + q^{\sum_{m'} \widetilde{x}^{m'} + x^m \mathbf{1}_0(\widetilde{x}^m)}} & \text{, if } \tilde{x}^n = 0. \end{cases} \tag{20d}$$

Theorem III.3. *If a solution to the FP equation* (13) *exists, it can be found through a finite-dimensional fixed-point equation. Specifically, the solution can be found through the fixed-point equation* (20).

Proof: The result follows by showing that π can be computed using $\widetilde{x}$. This makes the dimension of the equations finite since each of the N strategies is a function of $\tilde{x}$ and b which are discrete-valued vectors with 6^N possible values.

In (13a) and (13b) we use

$$\pi\left(1 \,|\, x^n\right) = \frac{1}{1 + \left(\frac{p}{1-p}\right)^{\sum_m \widetilde{x}^m + x^n \mathbf{1}_0(\widetilde{x}^n)}} \tag{21}$$

and in (13b) we also use

$$\pi\left(x^m | v\right) = \begin{cases} \mathbf{1}_{\tilde{x}^m}\left(x^m\right) & \widetilde{x}^m \neq 0 \\ Q\left(x^m | v\right) & \widetilde{x}^m = 0 \end{cases} \tag{22}$$

■

IV. Computing a PBE though a quadratic-dimensional fixed-point equation

In this section, we exploit the structure of the FP equation (20) to further simplify the equations such that they have a polynomial dimension in N. For ease of presentation, we concentrate on the case of homogeneous players, where $\delta_n = \delta$ for all n. However, the results we present can be generalized to non-homogeneous players (with different δ's). The key for this simplification is the fact that the indexing of the players has no effect on the future reward a player estimates he would get by waiting. Since $\widetilde{x}$ contains this information, it can be reduced to the following two sums.

Definition IV.1. Define the aggregated state information as

$$y_t = \sum_{n=1}^{N} \widetilde{x}_t^n \in \mathcal{Y}. \tag{23}$$

Further, define the indicator that player n has revealed his private information as $r_t^n = |\tilde{x}_t^n|$. Using $z_t^n = \max\{r_t^n, b_t^n\}$, define the number of players who cannot reveal their private information after turn t by

$$w_t = \sum_{n=1}^{N} z_t^n \in \mathcal{W}. \tag{24}$$

We define the value function of the acting player $U_a : \mathcal{X} \times \mathcal{W} \times \mathcal{Y} \times \{0,1\}^2 \to \mathbb{R}$, where $U_a(x, w, y, r, b)$ means that his private information is $x^n = x$, he revealed if $r = 1$ and bought if $b = 1$, and w players cannot reveal any longer and the aggregated state information is y. Similarly, we define the value function of the non-acting player $U_{na}^{\tilde{r},b} : \mathcal{X} \times \mathcal{W} \times \mathcal{Y} \times \{0,1\} \to \mathbb{R}\ \forall \tilde{r}, b \in \{0,1\}^2$, where $U_{na}^{\tilde{r},\tilde{b}}(x, w, y, z)$ means that his private information is $x^m = x$, he revealed if $\tilde{r} = 1$ bought if $\tilde{b} = 1$, with an acting agent n who can reveal his private information if $z = 0$, and w, y as before.

Finally, define the update functions of the next player parameters G^z, G^{rb} and the next population parameters E and $EG^z = (E, G^z), EG^{rb} = \left(E, G^{rb}\right)$ as follows

$$G^{rb}(r,b,\gamma,a) = \begin{cases} 1, a & , b = 0 \text{ and } \gamma = \boldsymbol{I} \\ r, a & , b = 0 \text{ and } \gamma \neq \boldsymbol{I} \\ r, b & , \text{else} \end{cases} \tag{25}$$

$$G^z(z,\gamma) = \begin{cases} 0 & , \text{if } z = 0 \text{ and } \gamma = \mathbf{0} \\ 1 & , \text{else} \end{cases} \tag{26}$$

$$E(z, w, y, \gamma, a) = \begin{cases} (w, y) & z = 1 \text{ or } (z = 0, \gamma = \mathbf{0}) \\ (w+1, y) & z = 0 \text{ and } \gamma = \mathbf{1} \\ (w+1, y+2a-1) & z = 0 \text{ and } \gamma = \boldsymbol{I} \end{cases} \tag{27}$$

We consider the alternative FP equation (28). The next Theorem shows that by finding a solution to (28), we obtain a solution to the FP equation (20). Since the equations (28) have a polynomial (quadratic) dimension in N, this significantly reduces the complexity of solving (20).

Polynomial (Quadratic) Dimension Fixed-Point Equation

Denote $q = \frac{1-p}{p}$. For every $n \in \mathcal{N}$, $w \in \mathcal{W}$, $y \in \mathcal{Y}$, $r, b \in \{0,1\}$ we evaluate $\gamma^* = \phi\left[w, y, r, b\right]$ as follows

- If $b = 1$ then $\gamma^* = \mathbf{0}$.
- If $b = 0$ then γ^* is the solution of $\gamma^*(x) = \arg\max \left\{ \underbrace{A}_{\text{don't buy}}, \underbrace{\frac{q^{y+x^n \mathbf{1}_0(r^n)} - 1}{q^{y+x^n \mathbf{1}_0(r^n)} + 1}}_{\text{buy}} \right\}$ $\forall x \in \mathcal{X}$, where

$$A = \frac{\delta}{N} U_a\left(x^n, EG^{rb}\left(r^n, b^n, w, y, \gamma^*, 0\right)\right) + \frac{\delta}{N}\left(N - w - 1 + z^n\right) U_{na}^{G^{rb}(r,b,\gamma^*,0)}\left(x^n, E\left(z^n, w, y, \gamma^*, 0\right), 0\right) + \frac{\delta}{N}\left(w - z^n\right) U_{na}^{G^{rb}(r^n,b^n,\gamma^*,0)}\left(x^n, E\left(z^n, w, y, \gamma^*, 0\right), 1\right) \quad \text{(28a)}$$

where $z^n = \max\left\{r^n, b^n\right\}$ and the value functions satisfy

$$U_a\left(x, w, y, r, b\right) = \begin{cases} 0 & b = 1 \\ A & b = 0, \gamma^*(x) = 0 \\ \frac{q^{y+x^n \mathbf{1}_0(r^n)} - 1}{q^{y+x^n \mathbf{1}_0(r^n)} + 1} & b = 0, \gamma^*(x) = 1 \end{cases} \quad \text{(28b)}$$

and for all $\widetilde{r}, \tilde{b}$ if $\tilde{b} = 1$ then $U_{na}^{\widetilde{r},\tilde{b}}\left(x, w, y, z\right) = 0$ and if $\tilde{b} = 0$

$$U_{na}^{\widetilde{r},\tilde{b}}\left(x, w, y, z\right) = \frac{\delta}{N}\mathbb{E}\left\{U_a\left(x, E\left(z, w, y, \gamma^*, \gamma^*\left(X^n\right)\right), \widetilde{r}, \tilde{b}\right)\right\} + \frac{\delta}{N}\mathbb{E}\left\{U_{na}^{\widetilde{r},\tilde{b}}\left(x, EG^z\left(z, w, y, \gamma^*, \gamma^*\left(X^n\right)\right)\right)\right\} + \frac{\delta}{N}\left(w - z - \tilde{z}\right)\mathbb{E}\left\{U_{na}^{\widetilde{r},\tilde{b}}\left(x, EG^z\left(1, w, y, \gamma^*, \gamma^*\left(X^n\right)\right)\right)\right\} + \frac{\delta}{N}\left(N - w - 2 + z + \tilde{z}\right)\mathbb{E}\left\{U_{na}^{\widetilde{r},\tilde{b}}\left(x, EG^z\left(0, w, y, \gamma^*, \gamma^*\left(X^n\right)\right)\right)\right\} \quad \text{(28c)}$$

where $\tilde{z} = \max\{\widetilde{r}, \tilde{b}\}$ and the expectation in the last equation is wrt the RV X^n:

- If $z = 0$ then $\Pr(X^n = x^n | \widetilde{r}, \tilde{b}, x^m, w, y) = \frac{Q(x^n|-1) + Q(x^n|1) q^{y+x^m \mathbf{1}_0(\tilde{r})}}{q^{y+x^m \mathbf{1}_0(\tilde{r})} + 1}$.
- if $z = 1$ then the expectation degenerates since for either $X^n = -1$ or $X^n = 1$ (i.e., with probability 1) $U_{na}^{\widetilde{r},\tilde{b}}\left(x, EG^z\left(1, w, y, \gamma^*, \gamma^*\left(X^n\right)\right)\right) = U_{na}^{\widetilde{r},\tilde{b}}\left(x, w, y, z\right)$.

Specifically, given the solution U^* of (28) (together with ϕ) we construct the following strategies and value functions.

$$\gamma^* = \theta\left[n, \tilde{x}, b\right] = \phi\left[w, y, r^n, b^n\right] \quad \text{(29)}$$

$$\tilde{V}^m\left(\cdot, n, \tilde{x}, b\right) = \begin{cases} U_a\left(\cdot, w, y, \left|\tilde{x}^n\right|, b^n\right), & m = n \\ U_{na}^{\left|\tilde{x}^m\right|, b^m}\left(\cdot, w, y, z^n\right), & m \neq n \end{cases} \quad \text{(30)}$$

where note that w, y, r^n, b^n are all determined by $\tilde{x}, b$ by (23) and (24). We will show that these value functions are solutions of the original FP equation (20).

Theorem IV.2. *The value functions $(\tilde{V}^m)_{m \in \mathcal{N}}$ together with the strategy mapping $\gamma^* = \phi\left[\cdot\right]$ satisfy the FP equation (20).*

Proof: See Appendix A. ■

V. Informational Cascades

In this section, we employ the results on computing a PBE to conclude that an informational cascade happens even in a non-myopic scenario. We focus on the case of homogeneous players with $\delta_n = \delta$ for all n, although similar arguments can be applied to the general case. For any given game with homogeneous players, our results allow to answer the question whether an informational cascade occurs in the game by solving equations with a quadratic number (in N) of variables. All that is required is for the FP equation (28) to have a solution so that a PBE exists. This way, we can investigate informational cascades even in large games.

Definition V.1. Let ϕ be a solution to the FP equation (28). Define the random variable

$$\rho_t = \begin{cases} 1 & \phi[w_t, y_t, r_t, b_t] = \boldsymbol{I} \text{ and } r_t = 0 \\ 0 & \text{else} \end{cases} \tag{31}$$

which indicates if the player that acts at turn t reveals his private information. Define the time of the i-th revealing by $\tau_i = \min\left\{ t \,|\, i = \sum_{t'=0}^{t} \rho_{t'} \right\}$.

The next lemma characterizes the reason why cascades still occur in a non-myopic scenario, and even relatively fast.

Lemma V.2. *Let ϕ be a solution to the FP* (28). *The induced process $\{Y_{T_i}\}_i$ is a Markov chain where there exist y_R, y_L such that for all $y_L < y_{\tau_i} < y_R$, if $Y_{T_{i+1}}$ exists then*

$$\Pr\left(Y_{T_{i+1}} = y' | Y_{T_i} = y\right) = \begin{cases} \frac{p+(1-p)q^y}{q^y+1} & y' = y+1 \\ \frac{1-p+pq^y}{q^y+1} & y' = y-1 \end{cases} \tag{32}$$

and y_L, y_R are absorbing states.

Proof: First we show the Markovianity of Y_{T_i}

$$\Pr\left(Y_{T_{i+1}} = y' | Y_{T_{0:i}} = y_{0:i}\right) = \Pr\left(Y_{T_i} + X^{N_{T_i}} = y' \,|\, Y_{T_{0:i}} = y_{0:i}\right) \tag{33a}$$

$$= \Pr\left(X^{N_{T_i}} = y' - y_i | Y_{T_{0:i}} = y_{0:i}\right) \tag{33b}$$

$$= \frac{Q(y'-y_i|0) + Q(y'-y_i|1)q^{y_i}}{q^{y_i}+1} \tag{33c}$$

$$= \Pr\left(Y_{T_{i+1}} = y' | Y_{T_i} = y_i\right). \tag{33d}$$

Now we characterize the absorbing states. Since $\delta < 1$, for $Y_{\max} = \left\lceil 1 + \log_q\left(\frac{1+\delta}{1-\delta}\right) \right\rceil$ we have

$$\frac{q^{Y_{\max}+x^n \mathbf{1}_0(\tilde{x}^n)} - 1}{q^{Y_{\max}+x^n \mathbf{1}_0(\tilde{x}^n)} + 1} > \delta > \delta U_a\left(x, w_{t+1}, y_{t+1}, r_{t+1}, b_{t+1}\right). \tag{34}$$

So either $y_R = Y_{\max}$ is absorbing or there exists a $y_R < Y_{\max}$ that is absorbing. In $Y_t = y_R$, all players, regardless of x_n, prefer to buy. Similarly, for $Y_{\min} = -2$ we have

$$\frac{q^{-1}-1}{q^{-1}+1} = 2p - 1 < 0 < \delta U_a\left(x, w_{t+1}, y_{t+1}, r_{t+1}, b_{t+1}\right) \tag{35}$$

So either $y_L = Y_{\min} = -2$ or $y_L = -1$ is absorbing. In $Y_t = y_L$, all players, regardless of x_n, prefer to wait. Hence, in $Y_t = y_L$ or $Y_t = y_R$ no more revealings occur and Y_t (and Y_{T_i}) remains constant for all $t' > t$ with probability 1. ■

The absorbing states of the Markov chain we defined above are informational cascades. However, an informational cascade that occurs after all or almost all player have revealed their private information is of little interest. The following theorem shows that this is far from being the case.

Theorem V.3. *If a solution ϕ to the FP* (28) *exists, then the probability that an informational cascade occurs in finite time approaches 1 as $N \to \infty$.*

Furthermore, let M_N be a sequence such that $\lim_{N\to\infty} \frac{M_N}{\sqrt{N}} = 0$ and $\lim_{N\to\infty} M_N = \infty$.

1) *The probability that less than M_N players have revealed their private information before the cascade occurred approaches 1 $N \to \infty$.*
2) *If, in addition, the solution is such that $\phi\left[w, y, r, b\right] = \mathbf{1}$ implies $\phi\left[\hat{w}, y, r, b\right] = \mathbf{1}$ for all $\hat{w} > w$, then the cascade happens in less than M_N turns with a probability that approaches 1 as $N \to \infty$.*

Proof: See Appendix A. ■

VI. Conclusions

We studied a Bayesian learning scenario with non-myopic players. Our scenario generalizes the classic myopic and sequential one-shot scenario where informational cascades were first reported. This generalization requires an intricate analysis of the PBE of the dynamic game. By introducing structured strategies, we constructed fixed-point equations that can be solved for the PBE in finite time. In particular, with homogeneous players, they have a polynomial dimension in the number of players N. This allows for evaluating informational cascades in large games.

We proved that if a solution exists, than an informational cascade where only a small portion of the information has been revealed happens with high probability for a large N. This suggests that informational cascades are a general phenomena that is not limited to myopic scenarios. We conjecture that informational cascade can be shown to happen in many other Bayesian learning scenarios.

Proving conditions under which a solution to our fixed-point equations exists remains an open problem. For a game with a large number of players, we believe that explicit solutions can be found and verified using our equations.

Appendix A
Proofs

A. Proofs of Lemma III.2

Proof: We have

$$\Pr\left(h^c_t|x,v\right) \underset{(a)}{=} \prod_{\tau=0}^{t-1} s^{n_\tau}_\tau\left(a_{t'}|x^{n_\tau},a_{0:\tau-1}\right) = \prod_{m=1}^{N}\prod_{\tau=0,n_\tau=m}^{t-1} s^{n_\tau}_\tau\left(a_\tau|x^m,a_{0:\tau-1}\right) \quad (36)$$

where (a) uses the fact that the strategy of each player is a function only of his private history. Hence

$$\begin{aligned}\Pr\left(x\,|\,v\,,h^c_t\right) &= \frac{Q\left(v\right)\Pr\left(x|v\right)\Pr\left(h^c_t|x,v\right)}{\sum_{x'}\Pr\left(v,h^c_t,x'\right)} = \\ &\frac{Q\left(v\right)\prod_{m=1}^{N}Q\left(x^m|v\right)\Pr\left(h^c_t|x,v\right)}{Q\left(v\right)\sum_{x'}\left(\prod_{m=1}^{N}Q\left(x'^m|v\right)\Pr\left(h^c_t|x',v\right)\right)} = \\ &\frac{\prod_{m=1}^{N}Q\left(x^m|v\right)\prod_{m=1}^{N}\prod_{\tau=0,n_\tau=m}^{t-1}s^{n_\tau}_\tau\left(a_\tau|x^m,a_{0:\tau-1}\right)}{\sum_{x'}\left(\prod_{m=1}^{N}Q\left(x'^m|v\right)\prod_{\tau=0,n_\tau=m}^{t-1}s^{n_\tau}_\tau\left(a_{t'}|x'^m,a_{0:\tau-1}\right)\right)} = \\ &\prod_{m=1}^{N}\frac{Q\left(x^m|v\right)\prod_{\tau=0,n_\tau=m}^{t-1}s^{n_\tau}_\tau\left(a_\tau|x^m,a_{0:\tau-1}\right)}{\sum_{x'^m}\left(Q\left(x'^m|v\right)\prod_{\tau=0,n_\tau=m}^{t-1}s^{n_\tau}_\tau\left(a_\tau|x'^m,a_{0:\tau-1}\right)\right)} = \\ &\prod_{m=1}^{N}\Pr\left(x^m\,|\,v\,,h^c_t\right)\end{aligned} \quad (37)$$

which means that the private information variables $\{x^m\}$ are conditionally independent given $v\,,h^c_t$.

Now observe that

$$\Pr\left(x^m\,|\,v\,,h^c_t\right) = \Pr\left(x^m\,|\,v\,,h^c_{t-1},a_t,n_{t+1}\right) = \frac{\Pr\left(a_t\,|\,v\,,h^c_{t-1},x^m\right)}{\Pr\left(a_t\,|\,v\,,h^c_{t-1}\right)}\Pr\left(x^m\,|\,v\,,h^c_{t-1}\right) \quad (38)$$

If $\gamma_t \neq \boldsymbol{I}$ or $m \neq n_t$, then player m did not reveal his private information in turn t. Hence

$$\begin{aligned}\frac{\Pr\left(a_t\,|\,v\,,h^c_{t-1},x^m\right)}{\Pr\left(a_t\,|\,v\,,h^c_{t-1}\right)} &= \\ &\frac{\sum_{x^{n_t}}\Pr\left(a_t\,|\,v\,,h^c_{t-1},x^m,x^{n_t}\right)\Pr\left(x^{n_t}\,|\,v\,,h^c_{t-1},x^m\right)}{\Pr\left(a_t\,|\,v\,,h^c_{t-1}\right)} = \\ &\frac{\sum_{x^{n_t}}\Pr\left(a_t\,|\,v\,,h^c_{t-1},x^{n_t}\right)\Pr\left(x^{n_t}\,|\,v\,,h^c_{t-1}\right)}{\Pr\left(a_t\,|\,v\,,h^c_{t-1}\right)} = 1.\end{aligned} \quad (39)$$

We arrive at the recursive equation

$$\Pr\left(x^n\,|\,v\,,h^c_t\right) = \begin{cases} \mathbf{1}_{\widetilde{x}^n_t}\left(x^n\right) & \gamma_t = \boldsymbol{I},\, n_t = n \\ \Pr\left(x^n\,|\,v\,,h^c_{t-1}\right) & o.w. \end{cases} \quad (40)$$

for which the solution is

$$\Pr\left(x^n\,|\,v\,,h^c_t\right) = \begin{cases} \mathbf{1}_{\widetilde{x}^n_t}\left(x^n\right) & \widetilde{x}^n_t \neq 0 \\ Q\left(x^n|v\right) & \widetilde{x}^n_t = 0 \end{cases} \quad (41)$$

If $\gamma_t \neq \boldsymbol{I}$ or $\widetilde{x}_t^{n_t} \neq 0$ then $\pi_{t+1}(1) = \pi_t(1)$. If instead $\gamma_t = \boldsymbol{I}$ and $\widetilde{x}_t^{n_t} = 0$ then the belief on v can be updated as

$$\begin{aligned}
&\pi_{t+1}(1) = \Pr(V = 1 | a_{0:t}, n_{0:t+1}) = \\
&\Pr(V = 1 | a_{0:t}, n_{0:t}) = \frac{\Pr(V = 1, a_t | a_{0:t-1}, n_{0:t})}{\Pr(a_t | a_{0:t-1}, n_{0:t})} = \\
&\frac{\Pr(a_t | V = 1, a_{0:t-1}, n_{0:t}) \Pr(V = 1 | a_{0:t-1}, n_{0:t})}{\Pr(a_t | a_{0:t-1}, n_{0:t})} = \\
&\frac{\Pr(a_t | V = 1, a_{0:t-1}, n_{0:t}) \pi_t(1)}{\Pr(a_t | V = 1, h_t^c) \pi_t(1) + \Pr(a_t | V = -1, h_t^c) \pi_t(-1)} \underset{(a)}{=} \\
&\frac{p^{1-a_t}(1-p)^{a_t} \pi_t(1)}{p^{1-a_t}(1-p)^{a_t} \pi_t(1) + p^{a_t}(1-p)^{1-a_t}(1 - \pi_t(1))} = \\
&\frac{1}{1 + \left(\frac{1-\pi_t(1)}{\pi_t(1)}\right)\left(\frac{p}{1-p}\right)^{2a_t - 1}}
\end{aligned} \tag{42}$$

where (a) follows since player n_t reveals his private information at turn t, so it must determine his action and

$$\Pr(a_t \,|\, v, h_t^c) = \begin{cases} p^{1-a_t}(1-p)^{a_t} & \gamma_t = \boldsymbol{I}, v = 1, \widetilde{x}_t^{n_t} = 0 \\ p^{a_t}(1-p)^{1-a_t} & \gamma_t = \boldsymbol{I},\, v = -1, \widetilde{x}_t^{n_t} = 0 \end{cases} \tag{43}$$

The initial condition for the recursive equation (42) is $\pi_0(1) = \Pr(v = 1) = \frac{1}{2}$. As can be verified by substituting, the solution is $\pi_t(1) = \frac{1}{1 + \left(\frac{p}{1-p}\right)^{\sum_n \widetilde{x}_t^n}}$, so the common belief is solely determined by $\sum_n \widetilde{x}_t^n$. ■

B. Proof of Theorem IV.2

Proof: It is clear that the second term in (13a) is exactly the same in the FP (28). Now consider an active user n with r and $b = 0$. Consider the first term in (13a). Define the update function

$$f(\tilde{x}^n, \gamma, a) = \begin{cases} 2a - 1 & \gamma = \boldsymbol{I},\, \tilde{x}^n = 0 \\ \tilde{x}^n & o.w. \end{cases} \tag{44}$$

where it follows from (15) that if $n = n_t$ then $\tilde{x}_{t+1}^n = f(\tilde{x}_t^n, \gamma_t, a_t)$. The new parameters of the active player n are $\hat{r} = |f(\tilde{x}^n, \gamma^*, 0)|$ and $\hat{b} = b = 0$ or $(\hat{r}, \hat{b}) = G^{rb}(r, b, \gamma^*, 0)$ and $(\hat{w}, \hat{y}) = E(z^n, w, y, \gamma^*, 0)$ where

$z^n = \max\{|\tilde{x}^n|, b^n\}$. The implication of the above is that the first term in (13a) will be

$$\sum_{n'=1}^{N} \tilde{V}^n\left(x^n, n', \tilde{x}^{-n} f(\tilde{x}^n, \gamma^*, 0), b^{-n}0\right) =$$

$$\tilde{V}^n\left(x^n, n, \tilde{x}^{-n} f\left(\tilde{x}^n, \gamma^*, 0\right), b^{-n}0\right) +$$

$$\sum_{n'=1, n'\neq n}^{N} \tilde{V}^n\left(x^n, n', \tilde{x}^{-n} f\left(\tilde{x}^n, \gamma^*, 0\right), b^{-n}0\right) =$$

$$U_a\left(x^n, \hat{w}, \hat{y}, \hat{r}, 0\right) + \sum_{n'=1, n'\neq n}^{N} U_{na}^{\hat{r},\hat{b}}\left(x^n, \hat{w}, \hat{y}, z^{n'}\right) =$$

$$U_a\left(x^n, \hat{w}, \hat{y}, \hat{r}, 0\right) + \sum_{n'=1, n'\neq n, z^{n'}=0}^{N} U_{na}^{\hat{r},\hat{b}}\left(x^n, \hat{w}, \hat{y}, 0\right) +$$

$$\sum_{n'=1, n'\neq n, z^{n'}=1}^{N} U_{na}^{\hat{r},\hat{b}}\left(x^n, \hat{w}, \hat{y}, 1\right) =$$

$$U_a\left(x^n, \hat{w}, \hat{y}, \hat{r}, 0\right) + \left(N - w - 1 + z^n\right) U_{na}^{\hat{r},\hat{b}}\left(x^n, \hat{w}, \hat{y}, 0\right)$$

$$+ \left(w - z^n\right) U_{na}^{\hat{r},\hat{b}}\left(x^n, \hat{w}, \hat{y}, 1\right) \quad (45)$$

A similar procedure can now be followed for the last summation appearing in (13c) with the only difference that we now isolate two terms from the sum over n', the acting player n and the player with index m. ■

C. *Proof of Theorem V.3*

Proof: If Y_t remains constant with probability one for all $t' > t$, then it is an informational cascade from definition since y_t sums all the revealed private information. Hence, the absorbing states of Y_{T_i} are informational cascades. We have shown that some $Y_{T_i} = y_L \geq Y_{\min}$ and $Y_{T_i} = y_R \leq Y_{\max}$ are absorbing. The values of both y_L, y_R are independent of N. The transition probabilities of Y_{T_i} are $\frac{p+(1-p)q^y}{q^y+1}$ for moving right and $\frac{1-p+pq^y}{q^y+1}$ for moving left, so they are also independent of N. We conclude that the distribution (specifically, expectation and variance) of the absorption time is independent of N. Hence, for large enough N, the probability that the absorption time is larger than M_N vanishes to zero. This absorption time is counted in the number of revealings i. We conclude that the probability that a cascade occurs before M_N revealings occur approaches 1 as $N \to \infty$.

Now assume that $\phi\left[w, y, r, b\right] = \mathbf{1}$ implies that $\phi\left[\hat{w}, y, r, b\right] = \mathbf{1}$ for $\hat{w} > w$. Denote the number of turns up to turn M_N where the acting player n_t has $r^{n_t} = 1$ or $b^{n_t} = 1$ by $\overline{R}\left(M_N\right)$, which is stochastically dominated by a binomial distributed variable with $p = \frac{M_N}{N}$ and M_N trials since

$$\Pr\left(y_{t+1} = y_t \,|\, w_t\right) = \frac{w_t}{N} \leq \frac{M_N}{N}. \quad (46)$$

Hence, we get from Theorem A.1.11 in [18] that for all $N > 0$

$$\Pr\left(\overline{R}\left(M_N\right) \leq \frac{M_N^2}{N} + \sqrt{2}\right) \leq e^{-\frac{N}{M_N^2} + \sqrt{2}\frac{N^2}{M_N^4}}. \quad (47)$$

Therefore, we conclude that with high probability, at least $M_N - 2$ of the first turns are of players with $z^n = 0$. Assume that in turn $t < M_N - 2$ the acting player did not reveal his private information.

- If he waited, then $w_{t+1} = w_t$ and $y_{t+1} = y_t$. The next player with $z^n = 0$ will also wait since he uses the same strategy $\gamma^* = \phi\left[w, y, r, b\right]$.
- If he bought, then $w_{t+1} = w_t + 1$ and $y_{t+1} = y_t$. The next acting player with $z^n = 0$ will also buy for $x_n = -1, 1$ (and not reveal) since $w_{t+1} > w_t$.

The same argument applies for all subsequent players with $z^n = 0$, and from definition to players with $z^n = 1$, so a cascade occurred. ■

References

[1] A. V. Banerjee, "A simple model of herd behavior," *The Quarterly Journal of Economics*, vol. 107, no. 3, pp. 797–817, 1992.

[2] S. Bikhchandani, D. Hirshleifer, and I. Welch, "A theory of fads, fashion, custom, and cultural change as informational cascades," *Journal of political Economy*, vol. 100, no. 5, pp. 992–1026, 1992.

[3] T. N. Le, V. G. Subramanian, and R. A. Berry, "The impact of observation and action errors on informational cascades," in *Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on*, 2014.

[4] I. H. Lee, "On the convergence of informational cascades," *Journal of Economic theory*, vol. 61, no. 2, pp. 395–411, 1993.

[5] B. Çelen and S. Kariv, "Observational learning under imperfect information," *Games and Economic Behavior*, vol. 47, no. 1, pp. 72–86, 2004.

[6] D. Acemoglu, M. A. Dahleh, I. Lobel, and A. Ozdaglar, "Bayesian learning in social networks," *The Review of Economic Studies*, vol. 78, no. 4, pp. 1201–1236, 2011.

[7] T. N. Le, V. G. Subramanian, and R. A. Berry, "Information cascades with noise," *IEEE Transactions on Signal and Information Processing over Networks*, vol. 3, no. 2, pp. 239–251, 2017.

[8] Y. Peres, M. Z. Racz, A. Sly, and I. Stuhl, "How fragile are information cascades?" arXiv preprint arXiv:1711.04024.

[9] A. Jadbabaie, E. Mossel, and M. A. Rahimian, "Bayesian group decisions: Algorithms and complexity," arXiv preprint arXiv:1705.04770.

[10] W. Hann-Caruthers, V. V. Martynov, and O. Tamuz, "The speed of sequential asymptotic learning," *Journal of Economic Theory*, vol. 173, pp. 383–409, 2018.

[11] K. Dasaratha, B. Golub, and N. Hak, "Bayesian social learning in a dynamic environment," available at SSRN: https://ssrn.com/abstract=3097505.

[12] D. Vasal and A. Anastasopoulos, "Decentralized Bayesian learning in dynamic games," arXiv preprint arXiv:1607.06847, 2016.

[13] ——, "A systematic process for evaluating structured perfect Bayesian equilibria in dynamic games with asymmetric information," in *American Control Conference (ACC)*, 2016.

[14] A. Sinha and A. Anastasopoulos, "Structured perfect Bayesian equilibrium in infinite horizon dynamic games with asymmetric information," in *Communication, Control, and Computing (Allerton), 2016 54th Annual Allerton Conference on*, 2016, pp. 256–263.

[15] D. Vasal, A. Sinha, and A. Anastasopoulos, "A systematic process for evaluating structured perfect Bayesian equilibria in dynamic games with asymmetric information," *IEEE Transactions on Automatic Control*, vol. PP, no. 99, pp. 1–1, 2018.

[16] H. Tavafoghi, "On design and analysis of cyber-physical systems with strategic agents," Ph.D. dissertation, University of Michigan, Ann Arbor, 2017.

[17] D. Vasal, "Sequential decomposition of dynamic games with asymmetric information and dependent states," arXiv preprint arXiv:1801.03459, 2018.

[18] N. Alon and J. H. Spencer, *The probabilistic method*. John Wiley & Sons, 2015.